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On best p-approximation from affine subspaces: asymptotic expansion

José María Quesada, Juan Martínez-Moreno, Juan Navas

Departamento de Matemáticas, Universidad de Jaén, Paraje las Lagunillas, Campus Universitario, 23701 Jaén, Spain

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Abstract In this paper we consider the problem of best approximation in $\ell_p(n)$, $1 < p \leqslant \infty$. If h_p, $1 < p < \infty$, denotes the best p-approximation of the element $h \in \mathbb{R}^n$ from a proper affine subspace K of $\mathbb{R}^n$, $h \notin K$, then $\lim_{p\to\infty} h_p = h_\infty^*$, where h_∞^* is a best uniform approximation of h from K, the so-called strict uniform approximation. Our aim is to prove that for all $r \in \mathbb{N}$ there are $\alpha_j \in \mathbb{R}^n$, $1 \leqslant j \leqslant r$, such that

$$h_p = h_\infty^* + \frac{\alpha_1}{p-1} + \frac{\alpha_2}{(p-1)^2} + \cdots + \frac{\alpha_r}{(p-1)^r} + \gamma_p^{(r)},$$

with $\gamma_p^{(r)} \in \mathbb{R}^n$ and $\|\gamma_p^{(r)}\| = \mathcal{O}(p^{-r-1})$. ***To cite this article: J.M. Quesada et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 1077–1082.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Comportement asymptotique des meilleures p-approximations sur un sous-espace affine

Résumé Dans cette Note on considére le probléme de meilleure approximation dans $\ell_p(n)$, $1 < p \leqslant \infty$. Si h_p, $1 < p < \infty$, désigne la meilleure p-approximation de $h \in \mathbb{R}^n$ par éléments d'un sous-espace affine K de $\mathbb{R}^n$, $h \notin K$, alors $\lim_{p\to\infty} h_p = h_\infty^*$, où h_∞^* est une meilleure approximation uniforme de h par éléments de K, appelée approximation uniforme stricte. Nous prouvons que h_p admet un développement asymptotique du type

$$h_p = h_\infty^* + \frac{\alpha_1}{p-1} + \frac{\alpha_2}{(p-1)^2} + \cdots + \frac{\alpha_r}{(p-1)^r} + \gamma_p^{(r)},$$

avec $\alpha_l \in \mathbb{R}^n$, $1 \leqslant l \leqslant r$, $\gamma_p^{(r)} \in \mathbb{R}^n$ et $\|\gamma_p^{(r)}\| = \mathcal{O}(p^{-r-1})$. ***Pour citer cet article : J.M. Quesada et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 1077–1082.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Les ℓ_p-normes sont définies par (1). Soit K un sous-ensemble de $\mathbb{R}^n$ et $h \in \mathbb{R}^n \setminus K$. Nous disons que $h_p \in K$, $1 \leqslant p \leqslant \infty$, est une meilleure p-approximation de h par éléments de K si $\|h_p - h\|_p \leqslant \|f - h\|_p$ pour tout $f \in K$.

Si K est un sous-ensemble fermé de $\mathbb{R}^n$, alors l'existence de h_p est garantie. Si en outre K est convexe and $1 < p < \infty$, alors la meilleure p-approximation est unique. Dans le case $p = \infty$ nous dirons que h_∞

E-mail address: jquesada@ujaen.es (J.M. Quesada).

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est une meilleure approximation uniforme de h par éléments de K. En général, il n'y a pas unicité de la meilleure approximation uniforme. En revanche, on peut définir une unique meilleure approximation uniforme « stricte », h_∞^*, [4,8].

Il est bien connu, [1,5,8], que si K est un sous-espace affine de $\mathbb{R}^n$, alors

$$\lim_{p\to\infty} h_p = h_\infty^*.$$

Dans la littérature, cette convergence est appelée algorithme de Pólya. Dans [2,5] on preuve qu'il existe $M > 0$ tel que $p\|h_p - h_\infty^*\| \leqslant M$ pour tout $p \geqslant 1$. Dans [6] les auteurs démontrent qu'il existe des constantes $L_1, L_2 > 0$ et $0 \leqslant a \leqslant 1$, dépendant de K, telles que $L_1 a^p \leqslant p\|h_p - h_\infty^*\| \leqslant L_2 a^p$ pour tout $p \geqslant 1$.

Sans perte de généralité nous assumerons $h = 0$. Dan ce qui suit K est supposé être un sous-espace affine de $\mathbb{R}^n$. Nous écrivons $K = h_\infty^* + \mathcal{V}$, où $\mathcal{V}$ est un sous-espace vectoriel de $\mathbb{R}^n$. Il est bien connu (*voir* par exemple [9]) que h_p, $1 < p < \infty$, est la meilleure p-approximation de 0 par éléments de K si et seulement si

$$\sum_{j=1}^n v(j)\bigl|h_p(j)\bigr|^{p-1}\operatorname{sgn}\bigl(h_p(j)\bigr) = 0 \quad \text{pour tout } v \in \mathcal{V}.$$

La formule de caractérisation ci-dessus et le lemme suivant sont les principaux arguments dans la démonstration de notre résult principal.

LEMME 0.1. – *Donnés $r \in \mathbb{N}$ et $a_l \in \mathbb{R}$, $1 \leqslant l \leqslant r$, il existe des nombres réels c_l, $1 \leqslant l \leqslant r$, avec $c_l = c_l(a_1, \ldots, a_l)$, tels que*

$$\left(1 + \frac{a_1}{p} + \cdots + \frac{a_r}{p^r}\right)^p = c_0 + \frac{c_1}{p} + \cdots + \frac{c_r}{p^r} + \mathcal{O}\left(\frac{1}{p^{r+1}}\right),$$

où $c_0 = e^{a_1}$.

THÉORÈME 0.1. – *Soit K un sous-espace affine de $\mathbb{R}^n$, $0 \notin \mathbb{K}$. Pour $1 < p < \infty$, soit h_p la meilleure p-approximation de 0 par éléments de K et h_∞^* l'approximation uniforme stricte. Alors, donné $r \in \mathbb{N}$, il existe des vecteurs $\alpha_l \in \mathbb{R}^n$, $1 \leqslant l \leqslant r$, tels que*

$$h_p = h_\infty^* + \frac{\alpha_1}{p-1} + \cdots + \frac{\alpha_r}{(p-1)^r} + \gamma_p^{(r)},$$

où $\gamma_p^{(r)} \in \mathbb{R}^n$ et $\|\gamma_p^{(r)}\| = \mathcal{O}(p^{-r-1})$.

1. Introduction

For $x = (x(1), x(2), \ldots, x(n)) \in \mathbb{R}^n$, the ℓ_p-norms, $1 \leqslant p \leqslant \infty$, are defined by

$$\|x\|_p = \left(\sum_{j=1}^n |x(j)|^p\right)^{1/p}, \quad 1 \leqslant p < \infty,$$

$$\|x\| := \|x\|_\infty = \max_{1\leqslant j\leqslant n}\bigl|x(j)\bigr|. \tag{1}$$

Let $K \neq \emptyset$ be a subset of $\mathbb{R}^n$. For $h \in \mathbb{R}^n \setminus K$ and $1 \leqslant p \leqslant \infty$ we say that $h_p \in K$ is a best p-approximation of h from K if

$$\|h_p - h\|_p \leqslant \|f - h\|_p \quad \text{for all } f \in K.$$

If K is a closed set of $\mathbb{R}^n$, then the existence of h_p is guaranteed. Moreover, there exists an unique best p-approximation if K is a closed convex set and $1 < p < \infty$. Throughout this paper, K denotes a proper affine subspace of $\mathbb{R}^n$. Without loss of generality we will assume that $h = 0$ and $0 \notin K$. It is well known (*see*, for instance, [9]) that h_p, $1 < p < \infty$, is the best p-approximation of 0 from K if and only if

$$\sum_{j=1}^{n} \big(h_p(j) - f(j)\big)\big|h_p(j)\big|^{p-1} \operatorname{sgn}\big(h_p(j)\big) = 0 \quad \text{for all } f \in K. \tag{2}$$

Writing $K = f_0 + \mathcal{V}$ for some $f_0 \in K$ and $\mathcal{V}$ a linear subspace of $\mathbb{R}^n$, then (2) is just equivalent to

$$\sum_{j=1}^{n} v(j)\big|h_p(j)\big|^{p-1} \operatorname{sgn}\big(h_p(j)\big) = 0 \quad \text{for all } v \in \mathcal{V}. \tag{3}$$

In the case $p = \infty$ we will say that h_∞ is a best uniform approximation of 0 from K. In general, the unicity of the best uniform approximation is not guaranteed. However, an unique "strict uniform approximation", h^*_∞, can be defined [4,8]. It is known, [1,5,8], that if K is an affine subspace of $\mathbb{R}^n$, then

$$\lim_{p\to\infty} h_p = h^*_\infty.$$

In the literature, the convergence above is called Pólya algorithm and occurs at a rate no worse than $1/p$, (*see* [2,5]). In [5,6] the authors prove that there are constants $L_1, L_2 > 0$ and $0 \leqslant a \leqslant 1$, depending on K, such that $L_1 a^p \leqslant p\,\|h_p - h^*_\infty\| \leqslant L_2 a^p$ for all $p > 1$.

The aim of this paper is to prove that the best p-approximation h_p, has an asymptotic expansion of the form

$$h_p = h^*_\infty + \frac{\alpha_1}{p-1} + \frac{\alpha_2}{(p-1)^2} + \cdots + \frac{\alpha_r}{(p-1)^r} + \gamma_p^{(r)},$$

for some $\alpha_j \in \mathbb{R}^n$, $1 \leqslant j \leqslant r$, $\gamma_p^{(r)} \in \mathbb{R}^n$ and $\|\gamma_p^{(r)}\| = \mathcal{O}(p^{-r-1})$.

2. Notation and preliminary results

Without loss of generality, we will assume that $\|h^*_\infty\| = 1$, $h^*_\infty(j) \geqslant 0$, $1 \leqslant j \leqslant n$, and that the coordinates of h^*_∞ are in decreasing ordering. Let $1 = d_1 > d_2 > \cdots > d_s \geqslant 0$ denote all the different values of $h^*_\infty(j)$, $1 \leqslant j \leqslant n$, and $\{J_l\}_{l=1}^s$ the partition of $J := \{1, 2, \ldots, n\}$ defined by $J_l := \{j \in J : h^*_\infty(j) = d_l\}$, $1 \leqslant l \leqslant s$. We henceforth put $s_0 = s$ if $d_s > 0$ and $s_0 = s - 1$ if $d_s = 0$.

We can write $K = h^*_\infty + \mathcal{V}$, where $\mathcal{V}$ is a proper linear subspace of $\mathbb{R}^n$. It is possible to choose a basis $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ of $\mathcal{V}$ and a partition $\{I_k\}_{k=1}^s$ of $I := \{1, 2, \ldots, m\}$ such that for all $i \in I_k$, $1 \leqslant k \leqslant s$,
(p1) $v_i(j) = 0$, $\forall j \in J_l$, $1 \leqslant l < k$,
(p2) $v_i(j) \neq 0$ for some $j \in J_k$.
The set of indices I_k could be empty for some k, $1 \leqslant k \leqslant s$. However, for simplicity of notation, we suppose that $I_k \neq \emptyset$ for $1 \leqslant k \leqslant s_0$, this involves no loss of generality.

We will use the following result [5,6].

THEOREM 2.1. – *In the conditions above, let*

$$a = \max_{1\leqslant l,k\leqslant r} \left\{ \frac{d_l}{d_k} : \sum_{j\in J_l} v_i(j) \neq 0 \quad \text{for some } i \in I_k \right\}, \tag{4}$$

where a is assumed to be 0 if $\sum_{j\in J_l} v_i(j) = 0$ for all $i \in I_k$, $1 \leqslant k, l \leqslant s_0$. Then there are $L_1, L_2 > 0$ such that

$$L_1 a^p \leqslant p\|h_p - h^*_\infty\| \leqslant L_2 a^p, \quad \forall p > 1.$$

LEMMA 2.2. – *If $\{x_p\}$ is a sequence of real numbers such that $p\,|x_p| \to 0$ as $p \to \infty$, then*

$$(1 + x_p)^p = 1 + p x_p + R_p,$$

with $R_p = \mathrm{o}(p|x_p|)$.

Proof. – The proof follows immediately from the application of Taylor's formula to the function $\varphi(z) = (1+z)^p$ at $z = 0$. □

In the next formula we use the following standard notation. Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N}$. If $\mathbf{r} = (r_1, r_2, \ldots, r_k) \in \mathbb{N}_0^k$ and $\mathbf{a} = \{a_j\}_{j \in \mathbb{N}}$ is a sequence of real numbers, then we define $|\mathbf{r}| := r_1 + r_2 + \cdots + r_k$, $\mathbf{r}! := r_1! r_2! \cdots r_k!$ and $\mathbf{a}^{\mathbf{r}} = a_1^{r_1} a_2^{r_2} \cdots a_k^{r_k}$. Also, for $i \in \mathbb{N}$, we denote $\mathcal{G}(k,i) := \{\mathbf{r} \in \mathbb{N}_0^k : \sum_{j=1}^k j r_j = i\}$.

Let $\mathbf{a} = \{a_j\}_{j \in \mathbb{N}}$ and $\mathbf{b} = \{b_j\}_{j \in \mathbb{N}}$ be two sequences of real numbers and $m, n \in \mathbb{N}$. An easy computation gives

$$f_{m,n}(z) := \sum_{j=1}^{n} b_j \left(\sum_{i=1}^{m} a_i z^i \right)^j = \sum_{i=1}^{mn} \sum_{r \in \mathcal{G}(m,i)} \frac{|\mathbf{r}|!}{\mathbf{r}!} b_{|\mathbf{r}|} \mathbf{a}^{\mathbf{r}} z^i.$$

Applying the formula above and the Rolle theorem we easily obtain the expansion of known functions. For example, taking $b_j = 1/j!$, $j = 1, 2, \ldots,$ we get

$$\exp\left[a_1 z + \cdots + a_k z^k\right] = 1 + \sum_{i=1}^{k} \sum_{r \in \mathcal{G}(k,i)} \frac{\mathbf{a}^{\mathbf{r}}}{\mathbf{r}!} z^i + \mathcal{O}\left(z^{k+1}\right).$$

Analogously, taking $b_j = (-1)^{j-1}/j$, $j = 1, 2, \ldots,$ we have

$$\frac{1}{z} \log\left(1 + a_1 z + \cdots + a_k z^k\right) = \sum_{i=1}^{k+1} \sum_{r \in \mathcal{G}(k,i)} \frac{(-1)^{|\mathbf{r}|-1}(|\mathbf{r}|-1)!}{\mathbf{r}!} \mathbf{a}^{\mathbf{r}} z^{i-1} + \mathcal{O}\left(z^{k+1}\right).$$

Now we could use the formulas above to obtain explicitly the asymptotic expansion of order k of the expression

$$\left(1 + \frac{a_1}{p} + \cdots + \frac{a_k}{p^k}\right)^p = \exp\left[p \log\left(1 + \frac{a_1}{p} + \cdots + \frac{a_k}{p^k}\right)\right].$$

However, in order to simplify the notation, we resume these observations in the following result.

LEMMA 2.3. – *Let $k \in \mathbb{N}$ and $a_l \in \mathbb{R}$, $1 \leqslant l \leqslant k$. Then there are $c_l \in \mathbb{R}$, $1 \leqslant l \leqslant k$, with $c_l = c_l(a_1, \ldots, a_l)$, such that*

$$\left(1 + \frac{a_1}{p} + \cdots + \frac{a_k}{p^k}\right)^p = c_0 + \frac{c_1}{p} + \cdots + \frac{c_k}{p^k} + \mathcal{O}\left(\frac{1}{p^{k+1}}\right),$$

where $c_0 = \mathrm{e}^{a_1}$.

For $v \in \mathcal{V}$, $v \neq 0$, and $1 \leqslant t \leqslant s$, let $J_t[v]$ be the set of indices j in J_t such that $v(j) \neq 0$ and define

$$t_v := \min\left\{t \in \{1, \ldots, s\} : J_t[v] \neq \emptyset\right\} \quad \text{and} \quad \widehat{J}_{t_v} := J_{t_v}[v].$$

Also, if $J' \subset J$ we denote by $\|\cdot\|_{J'}$ the restriction of the norm $\|\cdot\|$ to the set of indices in J'.

LEMMA 2.4. – *Suppose that there are $\alpha_l \in \mathcal{V}$, $1 \leqslant l \leqslant r$, such that*

$$h_p = h_\infty^* + \sum_{l=1}^{r} \frac{\alpha_l}{(p-1)^l} + \gamma_p^{(r)},$$

where $(p-1)^\tau \|\gamma_p^{(r)}\| \to 0$ as $p \to \infty$ for some $\tau \in \mathbb{N}$. Let $v \in \mathcal{V}$, $v \neq 0$, and suppose that $\|\alpha_l\|_{\widehat{J}_{t_v}} \neq 0$ for some $l \in \{0, 1, \ldots, r\}$, where $\alpha_0 := h_\infty^*$. Define

$$l_v := \min\left\{l \in \{0, 1, \ldots, r\} : \alpha_l(j) \neq 0 \text{ for some } j \in \widehat{J}_{t_v}\right\}$$

and let $\widehat{J}_{t_v}^0$ be the set of indices in $\widehat{J}_{t_v}$ such that $|\alpha_{l_v}(j)| = \|\alpha_{l_v}\|_{\widehat{J}_{t_v}}$. Then

$$\sum_{j \in \widehat{J}_{t_v}^0} v(j) c_l(j) \operatorname{sgn}\left(\alpha_{l_v}(j)\right) = 0, \quad 0 \leqslant l \leqslant \tau - l_v - 1, \tag{5}$$

where, for each $j \in \widehat{J}_{t_v}$*, the coefficients* $c_l(j)$ *are given by Lemma* 2.3 *with* $a_l = \alpha_{l+l_v}(j)/\alpha_{l_v}(j)$*,* $1 \leqslant l \leqslant r - l_v$ *and* $k = r - l_v$*.*

Proof. – Note that we can assume $\tau \geqslant l_v - 1$; otherwise the condition in (5) is empty. Since $v \in \mathcal{V}$, from (3), we have

$$\sum_{j \in J} v(j) \left| h_p(j) \right|^{p-1} \operatorname{sgn}\big(h_p(j)\big) = 0, \tag{6}$$

where

$$h_p(j) = h_\infty^*(j) + \sum_{l=1}^{r} \frac{\alpha_l(j)}{(p-1)^l} + \gamma_p^{(r)}(j), \quad 1 \leqslant j \leqslant n.$$

Since $(p-l)^\tau \|\gamma_p^{(r)}\| \to 0$ as $p \to \infty$, it is possible to apply Lemmas 2.2 and 2.3 to obtain an appropriate expansion of $|h_p(j)|^{p-1}$. Now, multiplying (6) by $(p-1)^l$, $0 \leqslant l \leqslant \tau - l_v - 1$, and taking limits as $p \to \infty$ we deduce (5). □

3. Asymptotic behaviour of best p-approximations

In [7] we prove the following result:

THEOREM 3.1. – *Let* K *be a proper affine subspace of* $\mathbb{R}^n$*,* $0 \notin \mathbb{K}$*. For* $1 < p < \infty$*, let* h_p *denote the best* p*-approximation of* 0 *from* K *and let* h_∞^* *be the strict uniform approximation. Then, for all* $r \in \mathbb{N}$*, there are* $\alpha_l \in \mathbb{R}^n$*,* $1 \leqslant l \leqslant r$*, such that*

$$h_p = h_\infty^* + \frac{\alpha_1}{p-1} + \cdots + \frac{\alpha_r}{(p-1)^r} + \gamma_p^{(r)}, \tag{7}$$

where $\gamma_p^{(r)} \in \mathbb{R}^n$ *and* $\|\gamma_p^{(r)}\| = \mathcal{O}(p^{-r-1})$*.*

Proof. – Since $p \|h_p - h_\infty^*\|$ is bounded, [2,5], the proof follows immediately by induction on r with the help of Lemmas 3.2 and 3.3. □

LEMMA 3.2. – *Under the same conditions of Theorem* 3.1*, let* $r \in \mathbb{N}$ *and suppose that there are* $\alpha_l \in \mathcal{V}$*,* $1 \leqslant l \leqslant r-1$*, such that*

$$h_p = h_\infty^* + \sum_{l=1}^{r-1} \frac{\alpha_l}{(p-1)^l} + \gamma_p^{(r-1)}.$$

If there exists $\alpha_r := \lim_{p \to \infty} (p-1)^r \gamma_p^{(r-1)}$*, then* $(p-1)^{r+1} \|\gamma_p^{(r)}\|$ *is bounded, where* $\gamma_p^{(r)} := \gamma_p^{(r-1)} - \alpha_r/(p-1)^r$*.*

LEMMA 3.3. – *Under the same conditions of Theorem* 3.1*, let* $r \in \mathbb{N}$ *and suppose that there are* $\alpha_l \in \mathcal{V}$*,* $1 \leqslant l \leqslant r-1$*, such that*

$$h_p = h_\infty^* + \sum_{l=1}^{r-1} \frac{\alpha_l}{(p-1)^l} + \gamma_p^{(r-1)}.$$

If $(p-1)^r \|\gamma_p^{(r-1)}\|$ *is bounded, then there exists* $\lim_{p \to \infty} (p-1)^r \gamma_p^{(r-1)} \in \mathcal{V}$*.*

Remark 1. – Let us observe that if $\sum_{j \in J_k} v_i(j) = 0$ for all $i \in I_k$ and all $1 \leqslant k \leqslant s_0$, then, from (4), $0 \leqslant a < 1$. In this case, as a consequence of Theorem 2.1, $p^l \|h_p - h_\infty^*\| \to 0$ for all $l \in \mathbb{N}$ and hence (7) holds immediately for $\alpha_l = 0 \in \mathbb{R}^n$, for all $l = 1, \ldots, r$. Therefore, in order to get non trivial expansions of h_p, we must assume that $\sum_{j \in J_k} v_i(j) \neq 0$ for some $i \in I_k$, $1 \leqslant k \leqslant s_0$.

Remark 2. – In [3], the authors suggest the asymptotic expansion,

$$h_p = h_\infty^* + \sum_{i=1}^{\infty} \frac{B_i}{p^i}\,.$$

They apply the series above to obtain good estimations of h_∞^* by means of extrapolation techniques. However, to our knowledge, there was not any proof of this formula.

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