C. R. Acad. Sci. Paris, Ser. I 334 (2002) 1089–1094

Équations aux dérivées partielles/*Partial Differential Equations*

A maximum principle for bounded solutions of the telegraph equation in space dimension three

Jean Mawhin [a], Rafael Ortega [b,1], Aureliano M. Robles-Pérez [b]

[a] Département de mathématique, Université Catholique de Louvain, 1348, Louvain-la-Neuve, Belgium

[b] Departamento de Matemática Aplicada, Facultad de Ciencias, Universidad de Granada, 18071, Granada, Spain

Received 14 April 2002; accepted 15 April 2002

Note presented by Haïm Brézis.

Abstract A maximum principle is proved for the weak solutions $u \in L^\infty(\mathbb{R} \times \mathbb{T}^3)$ of the telegraph equation $u_{tt} - \Delta_x u + cu_t + \lambda u = f(t,x)$, in space dimension three, when $c > 0$, $\lambda \in (0, c^2/4]$ and $f \in L^\infty(\mathbb{R} \times \mathbb{T}^3)$ (Theorem 1). The result is extended to a solution and a forcing belonging to a suitable space of bounded measures (Theorem 2). Those results provide a method of upper and lower solutions for the semilinear equation $u_{tt} - \Delta_x u + cu_t = F(t,x,u)$. ***To cite this article: J. Mawhin et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 1089–1094.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Un principe du maximum pour les solutions bornées de l'équation des télégraphistes en dimension spatiale trois

Résumé On démontre un principe du maximum pour les solutions faibles $u \in L^\infty(\mathbb{R} \times \mathbb{T}^3)$ de l'équation des télégraphistes $u_{tt} - \Delta_x u + cu_t + \lambda u = f(t,x)$ en dimension spatiale trois lorsque $c > 0$, $\lambda \in (0, c^2/4]$ et $f \in L^\infty(\mathbb{R} \times \mathbb{T}^3)$ (Théorème 1). Le résultat est étendu à une solution et un terme forçant appartenant à un certain espace de mesures bornées (Théorème 2). Ces résultats fournissent une méthode de sous- et sur-solutions pour l'équation semilinéaire $u_{tt} - \Delta_x u + cu_t = F(t,x,u)$. ***Pour citer cet article : J. Mawhin et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 1089–1094.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Soit $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ le cercle unité. Le résultat principal de cette Note est le Théorème 1.

THÉORÈME 1. – *Pour chaque* $\lambda \in (0, c^2/4]$ *et chaque* $f \in L^\infty(\mathbb{R} \times \mathbb{T}^3)$, *le problème*

$$\mathcal{L}u + \lambda u := u_{tt} - \Delta_x u + cu_t + \lambda u = f(t,x) \quad \textit{dans } \mathcal{D}'\big(\mathbb{R} \times \mathbb{T}^3\big),\ u \in L^\infty\big(\mathbb{R} \times \mathbb{T}^3\big)$$

a une solution unique. En outre, si $f \geqslant 0$ *p.p. in* $\mathbb{R} \times \mathbb{T}^3$, *alors* $u \geqslant 0$ *p.p. dans* $\mathbb{R} \times \mathbb{T}^3$.

E-mail addresses: mawhin@amm.ucl.ac.be (J. Mawhin); rortega@ugr.es (R. Ortega); arobles@ugr.es (A.M. Robles-Pérez).

S1631-073X(02)02406-8/FLA

L'existence et le principe du maximum se démontrent d'abord pour $\lambda = c^2/4$ en construisant une mesure positive $\mathcal{U}_3$ telle que $u = \mathcal{U}_3 * f$. Le cas où $\lambda \in (0, c^2/4)$ s'en déduit en écrivant l'équation sous la forme $\mathcal{L}u + (c^2/4)u = f(t,x) + [(c^2/4) - \lambda]u$. La condition sur λ est optimale.

Le théorème est étendu comme suit à des mesures bornées. On désigne par $\mathcal{E}$ l'ensemble des mesures μ sur $\mathbb{R} \times \mathbb{T}^3$ telles que l'ensemble E_μ défini par

$$\left\{\langle \mu, \phi\rangle : \phi \in C^0\left(\mathbb{R} \times \mathbb{T}^3\right),\ \|\phi\|_\infty = 1,\ \operatorname{supp}(\phi) \subset [h-\pi, h+\pi] \times \mathbb{T}^3,\ h \in \mathbb{R}\right\}$$

soit majoré. C'est un espace de Banach pour la norme $\|\mu\|_{\mathcal{E}} = \sup E_\mu$.

THÉORÈME 2. – *Pour chaque $\lambda \in (0, c^2/4]$ et chaque $\mu \in \mathcal{E}$, le problème*

$$\eta_{tt} - \Delta_x \eta + c\eta_t + \lambda\eta = \mu \quad \textit{dans } \mathcal{D}'\left(\mathbb{R} \times \mathbb{T}^3\right),\ \eta \in \mathcal{E},$$

a une solution unique. En outre, $\eta \geqslant 0$ si $\mu \geqslant 0$.

Pour démontrer le Théorème 2, on définit une norme sur $L^\infty(\mathbb{R} \times \mathbb{T}^3)$ qui coïncide avec celle de $\mathcal{E}$ pour les mesures du type $f\,dt\,dx$, et, dans ce cadre, on estime la norme de la solution u en termes de celle de f (Lemme 3). On approche alors μ, au sens de la convergence vague, par une suite de mesures $f_n\,dt\,dx$ avec $f_n \in L^\infty(\mathbb{R} \times \mathbb{T}^3)$, et on passe à la limite pour une sous-suite convenable.

Ces résultats permettent l'élaboration d'une méthode des sous- et des sur-solutions pour le problème

$$u_{tt} - \Delta_x u + cu_t = F(t,x,u) \quad \text{dans } \mathcal{D}'\left(\mathbb{R} \times \mathbb{T}^3\right),\ u \in L^\infty\left(\mathbb{R} \times \mathbb{T}^3\right)$$

lorsque $u \mapsto F(t,x,u) + (c^2/4)u$ est croissante.

1. Introduction

It was proved in [4] that a maximum principle holds for the weak doubly 2π-periodic solutions of the telegraph equation

$$u_{tt} - u_{xx} + cu_t + \lambda u = f(t,x) \quad (c > 0), \tag{LT}$$

if and only if $\lambda \in (0, \nu(c)]$, where $\nu(c)$ is some number contained in the interval $\left(\frac{c^2}{4}, \frac{c^2}{4} + \frac{1}{4}\right]$. This maximum principle on a torus was used in [4] to develope a method of upper and lower solutions for the weak doubly 2π-periodic solutions of semilinear telegraph equations of the form

$$u_{tt} - u_{xx} + cu_t = F(t,x,u), \tag{NT}$$

when the function $u \mapsto F(t,x,u) + \nu(c)u$ is nondecreasing.

In [3], those results were extended to the solutions $u(t,x)$ of Eq. (LT) which are 2π-periodic with respect to x and bounded over $\mathbb{R}$ with respect to t (namely $u \in L^\infty(\mathbb{R} \times \mathbb{T})$, with $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ the unit circle). It was shown that, if $\lambda \in (0, c^2/4]$ and $f \in L^\infty(\mathbb{R} \times \mathbb{T})$, a maximum principle holds for the solutions of Eq. (LT) in $L^\infty(\mathbb{R} \times \mathbb{T}^3)$. The constant $c^2/4$ is optimal and the maximum principle is not strong. Using an approximation argument, the maximum principle was generalized to the case where f is replaced by an element in a suitable class of measures, allowing an extension, to Eq. (NT) with $u \mapsto F(t,x,u) + (c^2/4)u$ nondecreasing, of the method of upper and lower solutions for the solutions in $L^\infty(\mathbb{R} \times \mathbb{T})$. This method was applied in [3] to various classes of semilinear equations (NT), including forced dissipative sine-Gordon equations. In some cases, the existence–uniqueness result together with an argument of Amerio's type [2] implied the existence of a unique almost periodic solution.

All those results were restricted to telegraph equations with space dimension one. The aim of this note is to present some extensions to the telegraph equation with space dimension three.

2. A maximum principle in $L^\infty(\mathbb{R}\times\mathbb{T}^3)$

DEFINITION 1. – Let $c>0$ and $f\in L^\infty(\mathbb{R}\times\mathbb{T}^3)$. A *bounded solution* of the problem

$$\mathcal{L}u+\lambda u := u_{tt}-\Delta_x u + cu_t + \lambda u = f(t,x) \quad \text{in } \mathbb{R}\times\mathbb{R}^3, \tag{1}$$
$$u(t,x_1+2\pi,x_2,x_3)=u(t,x_1,x_2+2\pi,x_3)=u(t,x_1,x_2,x_3+2\pi)=u(t,x)$$

is a function $u\in L^\infty(\mathbb{R}\times\mathbb{T}^3)$ satisfying

$$\int_{\mathbb{R}\times\mathbb{T}^3}(\mathcal{L}^*\phi+\lambda\phi)u=\int_{\mathbb{R}\times\mathbb{T}^3} f\phi \quad \text{for all } \phi\in\mathcal{D}\big(\mathbb{R}\times\mathbb{T}^3\big),$$

where $\mathcal{L}^*\phi=\phi_{tt}-\Delta\phi-c\phi_t$, i.e.

$$\mathcal{L}u+\lambda u=f \quad \text{in } \mathcal{D}'\big(\mathbb{R}\times\mathbb{T}^3\big),\ u\in L^\infty\big(\mathbb{R}\times\mathbb{T}^3\big). \tag{2}$$

THEOREM 1. – *For each* $\lambda\in(0,c^2/4]$ *and* $f\in L^\infty(\mathbb{R}\times\mathbb{T}^3)$, *problem* (2) *has a unique solution* u. *Moreover, if* $f\geqslant 0$ *a.e. in* $\mathbb{R}\times\mathbb{T}^3$, *then* $u\geqslant 0$ *a.e. in* $\mathbb{R}\times\mathbb{T}^3$.

Before proving Theorem 1, we make a few remarks about its statement.

Remark 1. – Remarks 1 and 2 in [3] prove that $c^2/4$ is optimal, and the maximum principle is not strong.

Remark 2. – By comparison with the case where f is constant, one deduces that the bounded solution of Eq. (1) satisfies

$$\|u\|_{L^\infty}\leqslant\frac{1}{\lambda}\|f\|_{L^\infty}.$$

Sketch of the proof of Theorem 1. – *Uniqueness.* By an indirect argument, let us assume that u is a nontrivial bounded solution of (2) for $f=0$. After convoluting u with an appropriate function $S\in\mathcal{D}(\mathbb{R}\times\mathbb{T}^3)$ we find another solution $v=u*S$ which is not identically zero and belongs to $L^\infty(\mathbb{R}\times\mathbb{T}^3)\cap C^\infty(\mathbb{R}\times\mathbb{T}^3)$. Let $\gamma=\gamma(x)$ be an eigenfunction of the Laplace–Beltrami operator on $\mathbb{T}^3$ such that

$$w(t)=\int_{\mathbb{T}^3} v(t,x)\gamma(x)\,\mathrm{d}x$$

is not identically zero. Since we are considering the flat metric on $\mathbb{T}^3$, γ is of the form $\gamma(x)=\mathrm{e}^{\mathrm{i}(\nu\cdot x)}$ for some $\nu\in\mathbb{Z}^3$. Now $w(t)$ is a bounded solution of

$$\ddot{v}+c\dot{v}+\eta v=0$$

for some positive constant η depending upon λ and the eigenvalue associated to γ, a contradiction.

Existence for $\lambda=c^2/4$. The function

$$\chi(x)=\frac{1}{4\pi|x|}\,\mathrm{e}^{-c|x|/2},\quad x\in\mathbb{R}^3\setminus\{0\},$$

belongs to $L^1(\mathbb{R}^3)$ and satisfies $\|\chi\|_{L^1(\mathbb{R}^3)}=4/c^2$. This allows us to define the positive measure $\mathcal{U}_3$ on $\mathbb{R}\times\mathbb{R}^3$ [1] given as

$$\langle\mathcal{U}_3,\phi\rangle=\frac{1}{4\pi}\int_{\mathbb{R}^3}\mathrm{e}^{-c|x|/2}\frac{\phi(|x|,x)}{|x|}\,\mathrm{d}x,\quad \phi\in C_0\big(\mathbb{R}\times\mathbb{R}^3\big).$$

Here $C_0(\mathbb{R}\times\mathbb{R}^3)$ is the space of continuous functions on $\mathbb{R}\times\mathbb{R}^3$ with compact support. It is easy to prove that $\|\mathcal{U}_3\|_{M(\mathbb{R}\times\mathbb{R}^3)}=4/c^2$. To prove that $\mathcal{U}_3$ is a solution of

$$\mathcal{L}u+\frac{c^2}{4}u=\delta \quad \text{in } \mathcal{D}'\big(\mathbb{R}\times\mathbb{R}^3\big),$$

with δ the Dirac measure, we use the following consequence of Kirchhoff's formula ([5], p. 177).

LEMMA 1. – *Let* $\phi \in \mathcal{D}(\mathbb{R}\times\mathbb{R}^3)$ *and* $\Box\phi = \phi_{tt} - \Delta_x\phi$. *Then*

$$\phi(0,0) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{(\Box\phi)(|x|,x)}{|x|}\,\mathrm{d}x.$$

Given $\phi \in \mathcal{D}(\mathbb{R}\times\mathbb{R}^3)$, define $\psi \in \mathcal{D}(\mathbb{R}\times\mathbb{R}^3)$ by $\psi(t,x) = \mathrm{e}^{-ct/2}\phi(t,x)$. Then $\mathcal{L}^*\phi + \frac{c^2}{4}\phi = \mathrm{e}^{ct/2}\Box\psi$ and so

$$\left\langle \left(\mathcal{L}+\frac{c^2}{4}\right)\mathcal{U}_3, \phi\right\rangle = \left\langle \mathcal{U}_3, \left(\mathcal{L}^*+\frac{c^2}{4}\right)\phi\right\rangle = \left\langle \mathcal{U}_3, \mathrm{e}^{ct/2}\Box\psi\right\rangle = \psi(0,0) = \phi(0,0).$$

Once we know that $\mathcal{U}_3$ is a fundamental solution of $\mathcal{L}+c^2/4$, for each $f \in \mathrm{L}^\infty(\mathbb{R}\times\mathbb{R}^3)$, the function u given by

$$u(t,x) = (\mathcal{U}_3 * f)(t,x) = \int_{\mathbb{R}\times\mathbb{R}^3} f(t-\tau, x-\xi)\,\mathrm{d}\mathcal{U}_3(\tau,\xi) \quad \text{a.e. in } \mathbb{R}\times\mathbb{R}^3$$

is well defined (*see* (14.9.2) of [1]), belongs to $\mathrm{L}^\infty(\mathbb{R}\times\mathbb{R}^3)$, and satisfies

$$\|u\|_{\mathrm{L}^\infty} \leqslant \|\mathcal{U}_3\|_M \|f\|_{\mathrm{L}^\infty} = \frac{4}{c^2}\|f\|_{\mathrm{L}^\infty}, \quad \mathcal{L}u + \frac{c^2}{4}u = f \quad \text{in } \mathcal{D}'\big(\mathbb{R}\times\mathbb{R}^3\big).$$

Assume now that $f \in \mathrm{L}^\infty(\mathbb{R}\times\mathbb{T}^3)$, so that $T_v f = f$ for each $v \in 2\pi\mathbb{Z}^3$, where $T_v f(t,x) = f(t,x+v)$. Then $T_v u = \mathcal{U}_3 * T_v f = \mathcal{U}_3 * f = u$ and so u is also in $\mathrm{L}^\infty(\mathbb{R}\times\mathbb{T}^3)$. To go from the case of test functions in $\mathcal{D}(\mathbb{R}\times\mathbb{R}^3)$ to test functions in $\mathcal{D}(\mathbb{R}\times\mathbb{T}^3)$ one uses, like in Lemma 5.1 of [4], a partition of unity periodic in x.

Existence for $\lambda \in (0, c^2/4)$. We proceed as in [3]. The linear operator

$$\mathcal{R} : \mathrm{L}^\infty\big(\mathbb{R}\times\mathbb{T}^3\big) \to \mathrm{L}^\infty\big(\mathbb{R}\times\mathbb{T}^3\big), \quad \mathcal{R}(f) = \mathcal{U}_3 * f$$

is bounded and has norm $4/c^2$. Problem (2) is equivalent to

$$\left[I - \left(\frac{c^2}{4}-\lambda\right)\mathcal{R}\right]u = \mathcal{R}f, \quad u \in \mathrm{L}^\infty\big(\mathbb{R}\times\mathbb{T}^3\big).$$

Since $\|(c^2/4-\lambda)\mathcal{R}\| < 1$, the inverse is given by

$$\left[I - \left(\frac{c^2}{4}-\lambda\right)\mathcal{R}\right]^{-1} = \sum_{n=0}^{\infty}\left(\frac{c^2}{4}-\lambda\right)^n \mathcal{R}^n.$$

This leads to the existence and positivity principles.

3. A maximum principle for some bounded measures

As in [3] we consider, for each $h \in \mathbb{R}$ and $K_h = [h-\pi, h+\pi]\times\mathbb{T}^3$, the seminorm in the space of measures $M(\mathbb{R}\times\mathbb{T}^3)$ defined as

$$\|\mu\|_h = \sup\big\{\langle\mu,\phi\rangle : \phi \in \mathrm{C}_0\big(\mathbb{R}\times\mathbb{T}^3\big),\ \|\phi\|_{\mathrm{L}^\infty} = 1,\ \mathrm{supp}(\phi) \subset K_h\big\}.$$

The class of measures for which this family of seminorms is bounded independently of h is denoted by $\mathcal{E}$. It becomes a Banach space with the norm

$$\|\mu\|_{\mathcal{E}} = \sup_{h\in\mathbb{R}}\|\mu\|_h.$$

We shall employ some properties of the vague topology in $M(\mathbb{R}\times\mathbb{T}^3)$ (*see* [1]).

LEMMA 2. – *Any bounded sequence* (μ_n) *in* $\mathcal{E}$ $(\sup_n \|\mu_n\|_{\mathcal{E}} \leqslant C < \infty)$ *contains a subsequence* μ_{n_k} *which converges vaguely to some* $\mu \in \mathcal{E}$.

As in [3] we define a norm in $L^\infty(\mathbb{R}\times\mathbb{T}^3)$ by

$$|||f||| = \sup_{h\in\mathbb{R}} \int_{K_h} \left|f(t,x)\right| \mathrm{d}t\,\mathrm{d}x = \|f\,\mathrm{d}t\,\mathrm{d}x\|_{\mathcal{E}}.$$

Following the proof of Lemma 1 in [3], given $\mu\in\mathcal{E}$ we find a sequence $f_n\in L^\infty(\mathbb{R}\times\mathbb{T}^3)$ such that $f_n\,\mathrm{d}t\,\mathrm{d}x$ converges vaguely to μ and $|||f_n|||\leqslant 2\|\mu\|_{\mathcal{E}}$. Moreover, if $\mu\geqslant 0$ then $f_n\geqslant 0$ a.e. in $\mathbb{R}\times\mathbb{T}^3$.

LEMMA 3. – *The solution* u *of problem* (2) *satisfies the inequality*

$$|||u||| \leqslant \frac{1}{\lambda}|||f|||.$$

Sketch of the proof. – *Case* $\lambda = c^2/4$. From $u=\mathcal{U}_3 * f$ we deduce that

$$u(t,x) = \int_{\mathbb{R}^3} \frac{\mathrm{e}^{-\frac{c}{2}|\xi|}}{4\pi|\xi|} f\left(t-|\xi|, x-\xi\right)\mathrm{d}\xi.$$

The result follows easily, as the periodicity of u with respect to x implies that

$$\int_{K_h}\left|f\left(t-|\xi|,x-\xi\right)\right|\mathrm{d}t\,\mathrm{d}x = \int_{K_h}\left|f\left(t-|\xi|,x\right)\right|\mathrm{d}t\,\mathrm{d}x = \int_{K_{h-|\xi|}}\left|f(t,x)\right|\mathrm{d}t\,\mathrm{d}x \leqslant |||f|||.$$

Case $\lambda\in(0,c^2/4)$. We employ the formula

$$u = \sum_{n=0}^{\infty}\left(\frac{c^2}{4}-\lambda\right)^n \mathcal{R}^{n+1} f,$$

where $\mathcal{R}f = \mathcal{U}_3 * f$ (convolution in $\mathbb{R}\times\mathbb{R}^3$). Then

$$|||u||| \leqslant \left[\sum_{n=0}^{\infty}\left(\frac{c^2}{4}-\lambda\right)^n\left(\frac{4}{c^2}\right)^{n+1}\right]|||f||| = \frac{1}{\lambda}|||f|||.$$

Given $\mu\in\mathcal{E}$ we consider the problem

$$\mathcal{L}\eta+\lambda\eta=\mu \quad \text{in } \mathcal{D}'\left(\mathbb{R}\times\mathbb{T}^3\right),\ \eta\in\mathcal{E}. \tag{3}$$

THEOREM 2. – *Problem* (3) *has a unique solution* η, *and* $\eta\geqslant 0$ *if* $\mu\geqslant 0$.

Sketch of the proof. – *Uniqueness.* Let $\eta\in\mathcal{E}$ be a solution of (3) with $\mu=0$. Given $S\in\mathcal{D}(\mathbb{R}\times\mathbb{T}^3)$ we consider $v=\eta * S$ (convolution in the topological group $\mathbb{R}\times\mathbb{T}^3$). Then $v=\eta*S$ is bounded and continuous (*see* [1], (14.9.3)) and, given $\phi\in\mathcal{D}(\mathbb{R}\times\mathbb{T}^3)$,

$$\int_{\mathbb{R}\times\mathbb{T}^3}(\mathcal{L}^*\phi)v = \int_{\mathbb{R}\times\mathbb{T}^3}(\mathcal{L}^*\phi)(\eta*S) = \langle\mathcal{L}^*\phi*\check{S},\eta\rangle = \left\langle\mathcal{L}^*(\phi*\check{S}),\eta\right\rangle$$
$$= -\lambda\langle\phi*\check{S},\eta\rangle = -\lambda\int_{\mathbb{R}\times\mathbb{T}^3}\phi(\eta*S) = -\lambda\int_{\mathbb{R}\times\mathbb{T}^3}\phi v,$$

where $\check{S}(t,x)=S(-t,-x)$. Thus $v=0$. Since S is arbitrary we conclude that also $\eta=0$.

Existence and positivity. We approximate $\mu\in\mathcal{E}$ (in the vague sense) by a sequence $f_n\in L^\infty(\mathbb{R}\times\mathbb{T}^3)$ with $|||f_n|||\leqslant 2\|\mu\|_{\mathcal{E}}$. The solution $u_n\in L^\infty(\mathbb{R}\times\mathbb{T}^3)$ of $\mathcal{L}u_n+\lambda u_n=f_n$ satisfies

$$|||u_n|||\leqslant\lambda^{-1}|||f_n|||\leqslant 2\lambda^{-1}\|\mu\|_{\mathcal{E}}.$$

The sequence $(\eta_n)=(u_n\,\mathrm{d}t\,\mathrm{d}x)$ contains a subsequence (η_{n_k}) vaguely convergent to some $\eta\in\mathcal{E}$. Since

$$\int_{\mathbb{R}\times\mathbb{T}^3} f_n\phi\to\langle\mu,\phi\rangle \quad\text{and}\quad \int_{\mathbb{R}\times\mathbb{T}^3} u_{n_k}\phi\to\langle\eta,\phi\rangle,$$

it is easy to conclude that η is a solution of (3). Also

$$\|\eta\|_{\mathcal{E}} \leqslant \sup \|\eta_n\|_{\mathcal{E}} \leqslant 2\lambda^{-1}\|\mu\|_{\mathcal{E}}.$$

Finally positivity is preserved by vague limits and this leads to the positivity principle.

Remark 3. – The results extend to doubly periodic solutions: if $f \in \mathrm{L}^\infty(\mathbb{T} \times \mathbb{T}^3)$, $u \in \mathrm{L}^\infty(\mathbb{T} \times \mathbb{T}^3)$.

This is just a consequence of the uniqueness of bounded solutions.

4. Final remark

The result about upper and lower solutions in Section 4 of [3] extends word by word. The only difference is the regularity of the solution, $u \in \mathrm{L}^\infty(\mathbb{R} \times \mathbb{T}^3)$. In one dimension we could prove $u \in \mathrm{W}^{1,\infty}(\mathbb{R} \times \mathbb{T})$. There is an analogue of Lemma 3 in [3], deduced from the maximum principle for measures. In going to the limit we do not need the compactness of $\{\underline{u}_n\}$ or $\{\overline{u}_n\}$. These sequences are bounded and monotone and one can use dominated convergence. A different matter seems to be the result about almost periodic solutions of [3]. Here the compactness inherited by the regularity $u \in \mathrm{W}^{1,\infty}(\mathbb{R} \times \mathbb{T})$ was important.

[1] Supported by DGES PB98-1294 (Spain).

References

[1] J. Dieudonné, Éléments d’analyse, Tome II, Gauthier-Villars, Paris, 1974.
[2] A.M. Fink, Almost Periodic Differential Equations, Lecture Notes in Math., Vol. 377, Springer, Berlin, 1974.
[3] J. Mawhin, R. Ortega, A.M. Robles-Pérez, A maximum principle for bounded solutions of the telegraph equations and applications to nonlinear forcings, J. Math. Anal. Appl. 251 (2000) 695–709.
[4] R. Ortega, A.M. Robles-Pérez, A maximum principle for periodic solutions of the telegraph equation, J. Math. Anal. Appl. 221 (1998) 625–651.
[5] V.S. Vladimirov, Equations of Mathematical Physics, Marcel Dekker, New York, 1971.