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Équations aux dérivées partielles/*Partial Differential Equations*

Solutions, concentrating on spheres, to symmetric singularly perturbed problems

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Abstract We discuss some existence results concerning problems (NLS) and (N), proving the existence of radial solutions concentrating on a sphere. ***To cite this article: A. Ambrosetti et al., C. R. Acad. Sci. Paris, Ser. I 335 (2002) 145–150.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Solutions concentreés sur spheres des problèmes de perturbation singulière

Résumé Nous étudions des problèmes de perturbations singulières (NLS), (N). On montre l'existence de solutions positives qui se concentrent sur une sphère. ***Pour citer cet article : A. Ambrosetti et al., C. R. Acad. Sci. Paris, Ser. I 335 (2002) 145–150.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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On considère le problème de perturbation singulière (NLS) et on suppose que $V \in C^1(\mathbb{R}^+, \mathbb{R})$ est bornée avec $\lambda_0^2 := \inf\{V(|x|) : x \in \mathbb{R}^n\} > 0$. On montre alors (*voir* Théorème 1) que pour tout $p > 1$, (NLS) admet une solution radiale qui se concentre atour de la sphère $\{|x| = \bar{r}\}$ où $\bar{r}$ est un maximum ou minimum local strict du potentiel auxiliarie $M(r) = r^{n-1}V^{\theta}(r)$, $\theta = (p+1)/(p-1) - 1/2$. De plus, génériquement, les solutions apparaissent on paires. On montre aussi que l'indice de Morse de ces solutions divèrge vers l'infini et ceci implique l'existence d'une infinité de solutions non radiales. D'autre part, si (NLS) admet une solution concentrée sur $\{|x| = \hat{r}\}$ alors on a nécessariement $M'(\hat{r}) = 0$ (*voir* Théorème 2).

La méthode de démonstration utilise une modification convenable de l'approche par perturbation, de nature variationelle, employée dans [4]. Un nouvel aspect important est la possibilité de localiser le problème et ceci nous permet de traiter également le cas d'exposants p critiques ou surcritques.

Des résultats semblables sont montrés pour le problème de Neumann (N).

Une version complète de ces résultats est contenue dans le travail [3].

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1. Introduction

Elliptic singularly perturbed problems such as

$$-\varepsilon^2\Delta u + V(x)u = u^p, \quad u \in \mathrm{H}^1\big(\mathbb{R}^n\big),\ u > 0, \tag{1}$$

have been extensively studied. Roughly, when V is a C^1 bounded potential with $\lambda_0^2 := \inf\{V(x) : x \in \mathbb{R}^n\} > 0$, and $1 < p < (n+2)/(n-2)$ (we assume here and in the sequel that $n \geqslant 3$), it has been shown that (1) possesses solutions concentrating near stationary points of V, see, e.g., [2] and references therein. See also [4] for multiplicity results when V has a manifold of stationary points. However all these papers deal with solutions, usually referred as *spikes*, concentrating at a single or at a finite number of points. Similar results hold for singularly perturbed boundary value problems with Neumann (or Dirichlet) boundary conditions, see, e.g., [7–9]. On the contrary, much less is known about the existence of solutions that concentrate at a higher dimensional manifold. In a recent paper [6] it has been proved that, in the case of the Neumann problem

$$\begin{cases} -\varepsilon^2\Delta u + u = u^p, & x \in \Omega,\\ \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{N}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain, there exist positive solutions concentrating on the boundary $\partial\Omega$ for some sequence $\varepsilon_n \to 0$. Moreover, the Morse index of these solutions becomes higher and higher. The extension of such a result to any bounded domain in $\mathbb{R}^n$ seems a matter of high technical difficulty. Though the problem of finding concentration on higher dimensional manifolds seem quite far from trivial, in the special case of radial symmetry some results can be achieved and they will be outlined in the present Note. For brevity, we will mainly focus on a problem (1). In Section 4 we will also outline some results dealing with (N). A complete version of the results sketched in this Note is contained in the forthcoming paper [3].

2. The main results

Let $V \in \mathrm{C}^1(\mathbb{R}^+, \mathbb{R})$ and $p > 1$ and consider the problem

$$-\varepsilon^2\Delta u + V\big(|x|\big)u = u^p, \quad u \in \mathrm{H}_r^1,\ u > 0, \tag{NLS}$$

where $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^n$ and $\mathrm{H}_r^1 = \mathrm{H}_r^1(\mathbb{R}^n)$ denotes the subspace of the radial functions in $\mathrm{H}^1(\mathbb{R}^n)$. If $1 < p < (n+2)/(n-2)$ the embedding of H_r^1 into L^p is compact and then a straight application of the Mountain–Pass theorem yields the existence of a radial solution of (NLS) concentrating on $x = 0$. Here we look for radial solutions concentrating on a sphere, without any limitation on the exponent $p > 1$.

In order to motivate our main existence result, let us take for the moment $p \in \,]1, (n+2)/(n-2)]$ and consider the C^2 functional $I_\varepsilon : \mathrm{H}_r^1 \to \mathbb{R}$,

$$\begin{aligned} I_\varepsilon(u) &= \frac{1}{2}\int_{\mathbb{R}^n}\big[|\nabla u|^2 + V\big(\varepsilon|x|\big)u^2\big]\,\mathrm{d}x - \frac{1}{p+1}\int_{\mathbb{R}^n}|u|^{p+1}\,\mathrm{d}x\\ &= \frac{1}{2}\int_0^{+\infty} r^{n-1}\big[(u')^2 + V(\varepsilon r)u^2\big]\,\mathrm{d}r - \frac{1}{p+1}\int_0^{+\infty} r^{n-1}|u|^{p+1}\,\mathrm{d}r. \end{aligned} \tag{2}$$

If u is a critical point of I_ε, then $-\Delta u + V(\varepsilon|x|)u = u^p$ and hence $u(x/\varepsilon)$ is a solution of (NLS). Roughly, in the energy integral I_ε one can distinguish two parts: a first one is the energy due to the potential V that would lead the possible radius of concentration r_0 tend to minima of V; the remainder part in I_ε is a *volume* *energy* and would lead $r_0 \to 0$. The existence of a solution concentrating at $|x| = r_0$ arises when the two effects balance each other, and this is quantified by an *auxiliary potential* M defined by setting

$$M(r) = r^{n-1} V^{\theta}(r), \quad \theta = \frac{p+1}{p-1} - \frac{1}{2}.$$

Different from the case of spikes, it is the function M, not just V, that plays a role in proving the existence of radial solutions of (NLS) concentrating on a sphere. Actually the following result holds:

THEOREM 1. – *Let $p > 1$, $V \in C^1(\mathbb{R}^+, \mathbb{R})$ be bounded and assume that $\lambda_0^2 := \inf\{V(|x|) : x \in \mathbb{R}^n\} > 0$. Moreover, suppose that M has a point of strict local maximum or minimum at $r = \overline{r}$. Then, for $\varepsilon > 0$ small enough,* (NLS) *has a radial solution which concentrates near the sphere $|x| = \overline{r}$.*

Remark. – In addition to the fact that V is substituted by M, another specific feature of solutions concentrating on a sphere is that we can take any $p > 1$. In fact it is known that spikes do not exist when p equals the critical Sobolev exponent $(n+2)/(n-2)$.

An outline of the proof is carried over in the next section. Here we point out a further new feature of our result, namely that, for a generic V as above, solutions arise in pairs. Actually, from the behavior of M one deduces the following:

COROLLARY 1. – *Suppose that, in addition to the assumptions of Theorem* 1, *there exists $r^* > 0$ such that*

$$(n-1)V(r^*) + \theta\, r^* V'(r^*) < 0. \tag{3}$$

Then (NLS) *has a pair of solutions concentrating on spheres.*

The preceding results neatly improve those of [5] where $1 < p < (n+2)/(n-2)$ is assumed, some involved assumptions (that we do not need here) on the behavior on V are made, no multiplicity results are given, and the radius of concentration is not established.

3. Outline of the proof

Although the proof is related to that of [4], several new ingredients are required here. We will first deal with the case $p \in]1, (n+2)/(n-2)]$ when the functional I_ε is well-defined on H^1_r. The general one will be handled by means of a suitable truncation argument and a priori estimates in L^∞. We let:

- $U_{\rho,\varepsilon}(r)$ denote the positive solution of $-U'' + V(\varepsilon\rho)U = U^p$, $U'(0) = 0$;
- $\phi_\varepsilon(r)$ denote a smooth non-decreasing cut-off function which vanishes in a neighborhood of the origin;
- $Z = \{z_{\rho,\varepsilon}(r) := \phi_\varepsilon(r)\, U_{\rho,\varepsilon}(r-\rho) : \rho \sim \varepsilon^{-1}\}$.

We look for critical points of I_ε in the form $u = z + w$ with $z \in Z$ and $w \perp T_z Z$. For this it suffices:

(i) for all $z \in Z$ to find $w(\rho,\varepsilon) \perp T_z Z$ such that $I'_\varepsilon(z+w) \in T_z Z$, namely $I'_\varepsilon(z+w) = \alpha z'$ for some $\alpha \in \mathbb{R}$; and next

(ii) to find ρ_ε such that, setting $\Phi_\varepsilon(\rho) = I_\varepsilon(z_{\rho,\varepsilon} + w_{\rho,\varepsilon})$, there holds $\Phi'_\varepsilon(\rho_\varepsilon) = 0$.

If (i) and (ii) hold then, according to general results (*see* [1]) $u_\varepsilon = z_{\rho_\varepsilon,\varepsilon} + w_{\rho_\varepsilon,\varepsilon}$ is a critical point of I_ε and hence is a solution of (NLS). As for step (i), one first shows that $I''_\varepsilon(z_{\rho,\varepsilon})$ is invertible for all ε small and $\rho \sim \varepsilon^{-1}$. Then the equation $I'_\varepsilon(z+w) = \alpha z'$ is equivalent to

$$w = S_\varepsilon(w) := -\left[I''_\varepsilon(z)\right]^{-1}\left(I'_\varepsilon(z) + I'_\varepsilon(z+w) - I'_\varepsilon(z) - I''_\varepsilon(z)[w] - \alpha z'\right).$$

In order to find fixed points of S_ε, some preliminary setting is in order. From the definition of $z_{\rho,\varepsilon}$ it follows that

$$z(r) \leqslant C\, e^{-\lambda_0|r-\rho|}.$$

Let $\eta > 0$ be such that

$$\lambda_1 := \lambda_0 - \eta > \frac{\lambda_0}{\min\{p, 2\}}$$

and define, for all $\gamma > 0$

$$\mathcal{C}_\varepsilon(\gamma) = \left\{ w \in \mathrm{H}^1_r : w \in (T_z Z)^\perp,\ \|w\|_{\mathrm{H}^1_r} \leqslant \gamma\varepsilon \|z\|_{\mathrm{H}^1_r},\ \left|w(r)\right| \leqslant \gamma\, \mathrm{e}^{-\lambda_1(\rho - r)} \text{ for } r \in [0, \rho] \right\}.$$

Using also the exponential decay of z one proves that, for $\varepsilon > 0$ small, $w \in \mathcal{C}_\varepsilon$ and $\rho \sim \varepsilon^{-1}$ one has $\|z_{\rho,\varepsilon}\|_{\mathrm{H}^1_r} \sim \varepsilon^{(1-n)/2}$. This, in turn, allows us to show the existence of $\gamma > 0$ such that:
(a) $w \in \mathcal{C}_\varepsilon(\gamma) \Rightarrow \|S_\varepsilon(w)\|_{\mathrm{H}^1_r} \leqslant \gamma\varepsilon \|z_{\rho,\varepsilon}\|_{\mathrm{H}^1_r}$;
(b) $w \in \mathcal{C}_\varepsilon(\gamma) \Rightarrow |(S_\varepsilon w)(r)| \leqslant \gamma\, \mathrm{e}^{-\lambda_1(\rho - r)}$ for $r \in [0, \rho]$.
In other words, S_ε maps $\mathcal{C}_\varepsilon(\gamma)$ into itself. Moreover, taking ε possibly smaller and, as before, $\rho \sim \varepsilon^{-1}$, S_ε is a contraction and therefore has a (unique) fixed point $w_{\rho,\varepsilon} \in \mathcal{C}_\varepsilon(\gamma)$.

About step (ii), one writes $I_\varepsilon(z_{\rho,\varepsilon} + w_{\rho,\varepsilon}) = I_\varepsilon(z_{\rho,\varepsilon}) + I'_\varepsilon(z_{\rho,\varepsilon})[w_{\rho,\varepsilon}] + \int I''_\varepsilon(z_{\rho,\varepsilon} + s w_{\rho,\varepsilon})[w_{\rho,\varepsilon}]^2\, \mathrm{d}s$. A straight forward calculation yields, for ε small, $\rho \sim \varepsilon^{-1}$ and $w \in \mathcal{C}_\varepsilon$,

$$\left\| I'_\varepsilon(z_{\rho,\varepsilon} + s w_{\rho,\varepsilon}) \right\| \sim \varepsilon^{(3-n)/2}, \qquad \left\| I''_\varepsilon(z_{\rho,\varepsilon} + s w_{\rho,\varepsilon}) \right\| \leqslant \mathit{const}.$$

Since, in addition, $\|w_{\rho,\varepsilon}\|_{\mathrm{H}^1_r} \leqslant \gamma\varepsilon \|z_{\rho,\varepsilon}\|_{\mathrm{H}^1_r}$ and $\|z_{\rho,\varepsilon}\|_{\mathrm{H}^1_r} \sim \varepsilon^{(1-n)/2}$, it follows that

$$I_\varepsilon(z_{\rho,\varepsilon} + w_{\rho,\varepsilon}) = I_\varepsilon(z_{\rho,\varepsilon}) + \mathrm{O}\left(\varepsilon^{3-n}\right).$$

By definition $z_{\rho,\varepsilon} = \phi_\varepsilon(r)\, U_{\rho,\varepsilon}(r - \rho)$ and this readily implies

$$I_\varepsilon(z_{\rho,\varepsilon}) \sim \rho^{n-1} \int \left(\left| U'_{\rho,\varepsilon}(r) \right|^2 + V(\varepsilon r) U^2_{\rho,\varepsilon}(r) - \frac{U^{p+1}_{\rho,\varepsilon}(r)}{p+1} \right) \mathrm{d}r.$$

Since $U_{\rho,\varepsilon}$ satisfies $-U'' + V(\varepsilon\rho)U = U^p$ then $U_{\rho,\varepsilon}(r) = \lambda^{2/(p-1)} U_1(\lambda r)$, where $\lambda^2 = V(\varepsilon\rho)$ and $-U''_1 + U_1 = U_1^p$. By a straight calculation, it follows that

$$\int \left(|U'_{\rho,\varepsilon}|^2 + V(\varepsilon r) U^2_{\rho,\varepsilon} - \frac{U^{p+1}_{\rho,\varepsilon}}{p+1} \right) \mathrm{d}r = c V^\theta(\varepsilon\rho), \quad c = \left(\frac{1}{2} - \frac{1}{p+1} \right) \int U_1^{p+1}.$$

In conclusion, for ε small and $\rho \sim \varepsilon^{-1}$ the following expansion holds true:

$$\Phi_\varepsilon(\rho) = I_\varepsilon(z_{\rho,\varepsilon} + w_{\rho,\varepsilon}) = \varepsilon^{1-n} \left[c M(\varepsilon\rho) + \mathrm{O}\left(\varepsilon^2\right) \right].$$

It follows that Φ_ε possesses a maximum (resp. minimum) at some $\rho_\varepsilon \sim \overline{r}/\varepsilon$, provided $\overline{r}$ is a maximum (resp. minimum) of M. This completes the proof of Theorem 1 when $1 < p \leqslant (n+2)/(n-2)$.

For $p > (n+2)/(n-2)$ and $M > 0$, we define a positive and smooth function $F_M : \mathbb{R} \to \mathbb{R}$ such that

$$F_M(t) = |t|^{p+1} \quad \text{for } |t| \leqslant M; \qquad F_M(t) = (M+1)^{p+1} \quad \text{for } |t| \geqslant M+1.$$

We also define the corresponding functional $I_{\varepsilon,M}$ on $\mathrm{H}^1(r)$ obtained substituting $|u|^{p+1}$ with $F_M(u)$ in I_ε. Define $M_0 = (\sup V)^{1/(p-1)}$. We note that, by definition, there holds $\|z_{\rho,\varepsilon}\|_\infty \leqslant M_0$ for all ρ and ε. Using this one can prove that if $M \geqslant M_0$, then (roughly) the operator $I''_\varepsilon(z)$ is invertible for all $z \in Z$, and the norm of its inverse is independent of M. Also, if $M \geqslant M_0 + \gamma$, the estimates involving $I'_\varepsilon(z+w)$ and $I''_\varepsilon(z+w)$, with $z \in Z$ and $w \in \mathcal{C}_\varepsilon(\gamma)$, are independent of M. In this way, one can chose γ depending only on p and V, and $M \geqslant M_0 + \gamma$ such that $S_{\varepsilon,M}$ (corresponding to $I''_{\varepsilon,M}$) is again a contraction on $\mathcal{C}_\varepsilon$.

4. Further results

As in the case of spikes, see [10], one can also prove a necessary condition for concentration on a sphere.

THEOREM 2. – *Suppose that, for all* $\varepsilon > 0$ *small,* (NLS) *has a radial solution* u_ε *concentrating on the sphere* $|x| = \widehat{r}$*, in the sense that* $\forall\delta > 0$, $\exists\varepsilon_0 > 0$ *and* $R > 0$ *such that*

$$u_\varepsilon(r) \leqslant \delta \quad \textit{for } \varepsilon \leqslant \varepsilon_0, \textit{ and } |r - \widehat{r}| \geqslant \varepsilon R. \tag{4}$$

Then u_ε *has a unique maximum* $r = r_\varepsilon$, $r_\varepsilon \to \widehat{r}$ *and* $M'(\widehat{r}) = 0$.

The next result shows that (NLS) has indeed many (non-radial) solutions in addition to the ones concentrating on spheres. Let u_ε denote the solutions found in Theorem 1.

THEOREM 3. – (i) *The solution* u_ε *obtained in Theorem* 1 *has the property that its Morse index, as critical point of the functional* I_ε *defined on all of* $\mathrm{H}^1(\mathbb{R}^n)$*, diverges as* $\varepsilon \downarrow 0$.

(ii) *If* V *is smooth and* $M''(\bar{u}) \neq 0$*, there exists* $\varepsilon_0 > 0$ *such that the set* $\{(\varepsilon, u_\varepsilon) : 0 < \varepsilon < \varepsilon_0\}$ *is a smooth curve in* $\mathbb{R} \times \mathrm{H}^1(\mathbb{R}^n)$ *and there exists a sequence* $\varepsilon_j \downarrow 0$ *such that from each* $u_{\varepsilon_j} \in \Lambda$ *bifurcates a family of non-radial solutions of* (NLS).

The same abstract method discussed before allows us to handle also problem (N), giving rise to a new class of concentrating solutions. Actually, we have the following result:

THEOREM 4. – *Let* Ω *be the annulus* $\{x \in \mathbb{R}^n : a < |x| < 1\}$*, where* $0 < a < 1$*. Then there exists a family of radial solutions* u_ε *of* (N) *concentrating at* $|x| = 1$*. More precisely* u_ε *possesses a local maximum point* $r_\varepsilon < 1$ *for which* $1 - r_\varepsilon \sim \varepsilon|\log\varepsilon|$.

Remarks. – (i) The result of Theorem 4 can be heuristically explained as follows. For problem (N), the boundary of Ω *attracts* the mass of solutions. In the case of spike-layers this determines their location depending on the geometry of Ω. In our case this attraction force balances the volume energy of the solution, which tends to shrink the circular crown.

(ii) The solutions u_ε concentrate at the boundary of the unit ball, although they have an interior maximum on the sphere $|x| = r_\varepsilon$. In other words, the profile of these solutions is that of an interior spike in one dimension, hence they are qualitatively different from those found in [6]. A natural and interesting result to pursue is to find solutions of (N) concentrating on an interior sphere or, for a general domain Ω, on an interior manifold.

(iii) Theorem 4 holds also when we prescribe Dirichlet boundary conditions. In this case one can prove analogous concentration results, at the inner boundary. Moreover, a concentration result like the one stated in Theorem 4 is also true when Ω is the unit ball.

(iv) The results in this paper can provide suggestions for more general results in non-symmetric cases, for which the above techniques do not apply. In [3] we will discuss some open problems and perspectives.

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References

[1] A. Ambrosetti, M. Badiale, Proc. Roy. Soc. Edinburg Sect. A 128 (1998) 1131–1161.
[2] A. Ambrosetti, M. Badiale, S. Cingolani, Arch. Rational Mech. Anal. 140 (1997) 285–300.
[3] A. Ambrosetti, A. Malchiodi, W.-M. Ni, Singularly perturbed elliptic equations with symmetry: Existence of solutions concentrating on spheres, to appear.
[4] A. Ambrosetti, A. Malchiodi, S. Secchi, Arch. Rational Mech. Anal. 159 (2001) 253–271.
[5] M. Badiale, T. D'Aprile, Concentration around a ssphere for a singularly perturbed Schrödinger equation, Preprint, Scuola Normale Superiore.

[6] A. Malchiodi, M. Montenegro, Boundary concentration phenomena for a singularly perturbed elliptic problem, Comm. Pure Appl. Math., to appear.
[7] W.-M. Ni, Notices Amer. Math. Soc. 45 (1) (1998) 9–18.
[8] W.-M. Ni, I. Takagi, Duke Math. J. 70 (1993) 247–281.
[9] W.-M. Ni, J. Wei, Comm. Pure Appl. Math. 48 (1995) 731–768.
[10] X. Wang, Comm. Math. Phys. 153 (1993) 229–243.