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Rational homotopy groups and Koszul algebras

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Abstract Let X and Y be finite-type CW-spaces (X connected, Y simply connected), such that the ring $H^*(Y,\mathbb{Q})$ is a k-rescaling of $H^*(X,\mathbb{Q})$. If $H^*(X,\mathbb{Q})$ is a Koszul algebra, then the graded Lie algebra $\pi_*(\Omega Y)\otimes\mathbb{Q}$ is the k-rescaling of $\mathrm{gr}_*(\pi_1 X)\otimes\mathbb{Q}$. If Y is a formal space, then the converse holds, and Y is coformal. Furthermore, if X is formal, with Koszul cohomology algebra, there exist filtered group isomorphisms between the Malcev completion of $\pi_1 X$, the completion of $[\Omega S^{2k+1}, \Omega Y]$, and the Milnor–Moore group of coalgebra maps from $H_*(\Omega S^{2k+1},\mathbb{Q})$ to $H_*(\Omega Y,\mathbb{Q})$. ***To cite this article: S. Papadima, A.I. Suciu, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 53–58.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Groupes d'homotopie rationnels et algèbres de Koszul

Résumé Soient X et Y deux CW-espaces de type fini (X connexe, Y simplement connexe), tels que l'anneau de cohomologie $H^*(Y,\mathbb{Q})$ soit un k-recalibrage de $H^*(X,\mathbb{Q})$. Si $H^*(X,\mathbb{Q})$ est une algèbre de Koszul, alors l'algèbre de Lie graduée $\pi_*(\Omega Y)\otimes\mathbb{Q}$ est le k-recalibrage de $\mathrm{gr}_*(\pi_1 X)\otimes\mathbb{Q}$. Si Y est un espace formel, alors l'implication réciproque est vraie aussi, et l'espace Y est coformel. De plus, si X est formel, avec algèbre de cohomologie de Koszul, on trouve des isomorphismes de groupes filtrés entre le complété de Malcev de $\pi_1 X$, le complété de $[\Omega S^{2k+1}, \Omega Y]$, et le groupe de Milnor–Moore d'applications de cogèbres entre $H_*(\Omega S^{2k+1},\mathbb{Q})$ et $H_*(\Omega Y,\mathbb{Q})$. ***Pour citer cet article : S. Papadima, A.I. Suciu, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 53–58.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Cette Note est un résumé des résultats de [9]. Commençons par définir la notion de recalibrage d'un espace topologique (ayant le type d'homotopie d'un CW-complexe connexe de type fini).

Soient k un entier positif, et A^* une algèbre graduée. On définit l'algèbre graduée $A[k]$ par $A[k]^{q(2k+1)} = A^q$ et $A[k]^p = 0$ si $2k+1 \nmid p$, avec la multiplication héritée de A. On dit qu'un espace Y est un *k-recalibrage* de X si $\pi_1 Y = 0$ et $H^*(Y,\mathbb{Q}) = H^*(X,\mathbb{Q})[k]$, en tant qu'algèbres graduées. Un tel espace Y peut être construit à partir du modèle minimal de l'algèbre $H^*(X,\mathbb{Q})[k]$, munie de la différentielle nulle. Cette construction donne un espace formel, mais X peut bien avoir des recalibrages non-formels. D'autre part, si $\dim_{\mathbb{Q}} H^*(X,\mathbb{Q}) < \infty$, alors X a un k-recalibrage unique (à $\mathbb{Q}$-équivalence près), pour tout $k \gg 1$.

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Soit L_* un espace vectoriel gradué, muni d'un crochet de Lie de degré 0. On définit l'algèbre de Lie graduée $L[k]$ par $L[k]_{2kq} = L_q$ et $L[k]_p = 0$ si $2k \nmid p$, avec le crochet hérité de L.

THÉORÈME 1. – *Soit Y un k-recalibrage d'un espace X. Soit $\mathrm{gr}_*(\pi_1 X) \otimes \mathbb{Q}$ l'espace vectoriel gradué associé à la suite centrale descendante de $\pi_1 X$ (muni du crochet induit par le commutateur du groupe), et soit $\pi_*(\Omega Y) \otimes \mathbb{Q}$ l'algèbre de Lie d'homotopie de Y (munie du crochet de Samelson).*

(a) *Si $A^* = H^*(X, \mathbb{Q})$ est une algèbre de Koszul (c'est-à-dire, si $\mathrm{Tor}^A_{p,q}(\mathbb{Q}, \mathbb{Q}) = 0$, pour tous $p \neq q$), alors il existe un isomorphisme d'algèbres de Lie graduées*

$$\pi_*(\Omega Y) \otimes \mathbb{Q} \cong \mathrm{gr}_*(\pi_1 X) \otimes \mathbb{Q}[k]. \tag{1}$$

De plus, $\prod_{i \geqslant 1} \left(1 - t^{(2k+1)i}\right)^{\mathrm{rank}\, \pi_{2ki}(\Omega Y)} = P_X(-t^{2k+1})$.

(b) *Si Y est formel et $\pi_*(\Omega Y) \otimes \mathbb{Q} \cong \mathrm{gr}_*(\pi_1 X) \otimes \mathbb{Q}[k]$, en tant qu'espaces vectoriels gradués, alors $H^*(X, \mathbb{Q})$ est une algèbre de Koszul. De plus, Y est coformel.*

La première partie du théorème nous permet de décrire le type d'homotopie rationnelle de l'espace de lacets de Y, uniquement à partir du polynôme de Poincaré de X. En particulier, $P_{\Omega Y}(t) = P_X(-t^{2k})^{-1}$.

THÉORÈME 2. – *Soit Y un k-recalibrage d'un espace X, tel que $H^*(X, \mathbb{Q})$ soit une algèbre de Koszul. L'espace X est formel si et seulement s'il existe des isomorphismes de groupes filtrés*

$$\mathrm{Hom}^{\text{cogèbre}}\left(H_*\left(\Omega S^{2k+1}, \mathbb{Q}\right), H_*(\Omega Y, \mathbb{Q})\right) \cong \left[\Omega S^{2k+1}, \Omega Y\right]^{\widehat{}} \cong \pi_1 X \otimes \mathbb{Q}. \tag{2}$$

Les groupes ci-dessus—le groupe de Milnor–Moore de morphismes de cogèbres entre $H_*(\Omega S^{2k+1}, \mathbb{Q})$ et $H_*(\Omega Y, \mathbb{Q})$, le complété du groupe de classes d'homotopie pointées entre ΩS^{2k+1} et ΩY, et le complété de Malcev du groupe fondamental de X—sont tous pourvus de filtrations canoniques de limite inverse. En passant aux gradués associés, l'isomorphisme de groupes filtrés $[\Omega S^{2k+1}, \Omega Y]^{\widehat{}} \cong \pi_1 X \otimes \mathbb{Q}$ donne l'isomorphisme d'algèbres de Lie graduées (1).

Parmi les espaces admettant des recalibrages intégraux, on trouve les compléments d'arrangements d'hyperplans dans $\mathbb{C}^\ell$, et les compléments d'entrelacs de cercles dans S^3. Le k-recalibrage (formel) d'un tel espace X est fourni par le complément Y d'un certain arrangement de sous-espaces de codimension $k+1$ dans $\mathbb{C}^{(k+1)\ell}$, et par le complément d'un certain entrelacs de $(2k+1)$-sphères dans S^{4k+3}, respectivement.

La formule (1) est vraie pour les arrangements supersolvables (un résultat de [2]), ainsi que pour les entrelacs ayant un graphe d'enlacement connexe. Pour les arrangements génériques, la formule (1) n'est plus vraie en général (à cause de la non-coformalité de Y, détectée par les produits de Whitehead d'ordre supérieur). Dans le cas des entrelacs, la formule (2) n'est pas toujours vraie (à cause de la non-formalité de X, détectée par les invariants de Campbell–Hausdorff), même si la formule (1) est valable.

1. Rescaling operations

This Note is an announcement of [9]. We refer to that paper for full details, and complete proofs.

Let A^* be a graded algebra over a ring R. For each integer $k \geqslant 1$, the *k-rescaling* of A is the graded algebra $A[k]$ with $A[k]^{q(2k+1)} = A^q$, and $A[k]^p = 0$ otherwise, and with multiplication rescaled accordingly.

Let X be a connected space. A simply-connected space Y is called a *k-rescaling* of X (over R) if the cohomology algebra $H^*(Y, R)$ is the k-rescaling of $H^*(X, R)$. For example, the sphere S^{2k+1} is a k-rescaling of S^1, the wedge $\bigvee^n S^{2k+1}$ is a k-rescaling of $\bigvee^n S^1$, and the connected sum $\#^g S^{2k+1} \times S^{2k+1}$ is a k-rescaling of a genus g orientable surface. Though here, and most throughout, the rescaling holds over $R = \mathbb{Z}$, the theory works best over $R = \mathbb{Q}$, and so this will be our default coefficients ring.

Using Sullivan's minimal models [14], it is easy to see that any connected CW-space of finite type, X, admits a rational k-rescaling, for each $k \geqslant 1$. Indeed, $(H^*(X, \mathbb{Q})[k], d = 0)$ is a 1-connected, finite-type differential graded algebra, with minimal model $\mathcal{M}$. Hence, there exists a finite-type, 1-connected CW-space Y such that $\mathcal{M}(Y) = \mathcal{M}$. In particular, $H^*(Y, \mathbb{Q}) = H^*(X, \mathbb{Q})[k]$.

By construction, the space Y is *formal*, i.e., its rational homotopy type is a formal consequence of its rational cohomology algebra. Hence, Y is uniquely determined (up to rational homotopy equivalence) among spaces with the same cohomology ring. But there may be other, non-formal rescalings of X. For example, take $X = S^1 \vee S^1 \vee S^{2k+2}$. Clearly, the formal k-rescaling is $Y = S^{2k+1} \vee S^{2k+1} \vee S^{(2k+1)(2k+2)}$. A non-formal rescaling is $Z = (S_x^{2k+1} \vee S_y^{2k+1}) \bigcup_\alpha e^{(2k+1)(2k+2)}$, where α is the iterated Whitehead product $\mathrm{ad}_x^{2k+2}(y) = [x, [\cdots [x, y]]]$. Even so, if $H^{>d}(X, \mathbb{Q}) = 0$, then X has a unique k-rescaling (up to $\mathbb{Q}$-equivalence), for all $k > (d-1)/2$; see [13].

A graded $\mathbb{Q}$-vector space L_*, endowed with a bilinear operation $[\ ,\ ]\colon L_p \otimes L_q \to L_{p+q}$ is called a *Lie algebra with grading* if the bracket satisfies the anti-commutativity and Jacobi identities. If the Lie identities are satisfied only up to sign (following the Koszul convention), then L_* is called a *graded Lie algebra*.

We are interested in two main examples. The *associated graded Lie algebra* of a finitely generated group G is the Lie algebra with grading $\mathrm{gr}_*(G) \otimes \mathbb{Q} := \bigoplus_{r \geqslant 1} (\Gamma_r G / \Gamma_{r+1} G) \otimes \mathbb{Q}$, where $\{\Gamma_r G\}_{r \geqslant 1}$ is the lower central series of G, and the bracket is induced by the group commutator. The *homotopy Lie algebra* of a based, simply-connected space Y is the graded Lie algebra $\pi_*(\Omega Y) \otimes \mathbb{Q} := \bigoplus_{r \geqslant 1} \pi_r(\Omega Y) \otimes \mathbb{Q}$, where ΩY is the loop space of Y, and the bracket is the Samelson product, obtained from the Whitehead product on $\pi_* Y$ via the boundary map in the path fibration over Y.

Given a Lie algebra with grading L_*, and a positive integer k, the *k-rescaling* of L is the graded Lie algebra $L[k]$, with $L[k]_{2kq} = L_q$ and $L[k]_p = 0$ otherwise, and with Lie bracket rescaled accordingly.

2. The Rescaling Formula

Let A^* be a connected, graded algebra over $\mathbb{Q}$. By definition, A is a *Koszul algebra* if $\mathrm{Tor}^A_{p,q}(\mathbb{Q}, \mathbb{Q}) = 0$, for all $p \neq q$. A necessary condition is that A be the quotient of a free algebra on generators in degree 1 by an ideal I generated in degree 2. A sufficient condition is that I admit a quadratic Gröbner basis. A topological interpretation of Koszulness is as follows. Let X be a formal space. Then, $H^*(X, \mathbb{Q})$ is a Koszul algebra if and only if the (Bousfield–Kan) rationalization $X_\mathbb{Q}$ is aspherical; see [10].

From now on, all spaces will be assumed to be connected, well-pointed, and homotopy equivalent to some finite-type CW-complex. Recall that a k-rescaling of a space X is a simply-connected space Y with $H^*(Y, \mathbb{Q}) = H^*(X, \mathbb{Q})[k]$, as graded algebras. Our first result shows that, under a Koszulness assumption, this homological rescaling passes to a homotopical rescaling.

THEOREM 2.1. – *Let Y be a k-rescaling of a space X. If $H^*(X, \mathbb{Q})$ is a Koszul algebra, then the following* Rescaling Formula *holds*:

$$\pi_*(\Omega Y) \otimes \mathbb{Q} \cong \mathrm{gr}_*(\pi_1 X) \otimes \mathbb{Q}[k], \quad \textit{as graded Lie algebras}. \tag{3}$$

We sketch the proof in the particular case when both X and Y are formal.

Let $A^* = H^*(X, \mathbb{Q})$, and let $\mathcal{H}_*(A) = \mathbb{L}^*(A_1)/(\mathrm{im}\,\nabla)$ be its holonomy Lie algebra, defined as the quotient of the free Lie algebra on the dual of A^1 by the Lie ideal generated by the image of the comultiplication map, $\nabla\colon A_2 \to A_1 \wedge A_1 = \mathbb{L}^2(A_1)$, and with grading given by bracket length. Since X is formal, $\mathrm{gr}_*(\pi_1 X) \otimes \mathbb{Q} \cong \mathcal{H}_*(A)$, as Lie algebras with grading (*see* for instance [6]).

Let $B^* = H^*(Y, \mathbb{Q}) = A^*[k]$, and let $\mathcal{L}(B, 0) = (\mathbb{L}(s^{-1}(\sharp B^{>0})), \partial)$ be the corresponding Quillen differential graded Lie algebra, defined as the free Lie algebra on the desuspension of the dual of the augmentation ideal of B, with differential ∂ arising from the dual of the multiplication map. Since Y is formal, $\pi_*(\Omega Y) \otimes \mathbb{Q} \cong H_*(\mathcal{L}(B, 0))$, as graded Lie algebras (*see* [11]).

Now define a morphism of graded Lie algebras, $\lambda\colon \mathcal{L}(B, 0) \to (\mathcal{H}_*(A)[k], 0)$, by sending A_1 identically to A_1 (in degree $2k$) and $A_{>1}$ to zero. It is readily checked that λ commutes with the differentials and induces a surjection in homology. Since the algebra A is Koszul, results from [11] and [10] insure that the induced map, $\lambda_*\colon \pi_*(\Omega Y) \otimes \mathbb{Q} \to \mathrm{gr}_*(\pi_1 X) \otimes \mathbb{Q}[k]$, is in fact an isomorphism (of graded Lie algebras).

Our next result shows that the Rescaling Formula (even at the level of graded vector spaces) is strong enough to imply—under a formality assumption—the Koszulness of $H^*(X, \mathbb{Q})$.

THEOREM 2.2. – *Let Y be a formal k-rescaling of a space X. If* $\mathrm{Hilb}(\pi_*(\Omega Y) \otimes \mathbb{Q}, t)$ *equals* $\mathrm{Hilb}(\mathrm{gr}_*(\pi_1 X) \otimes \mathbb{Q}, t^{2k})$*, then $H^*(X, \mathbb{Q})$ is a Koszul algebra. Moreover, Y is a* coformal *space* (*i.e., its rational homotopy type is determined by its homotopy Lie algebra*).

Let $P_X(t) = \mathrm{Hilb}(H^*(X; \mathbb{Q}), t)$ be the Poincaré series of X, and set $\phi_r = \mathrm{rank}\, \mathrm{gr}_r(\pi_1 X)$. The following *Lower Central Series formula* has received considerable attention: $\prod_{r \geqslant 1}(1 - t^r)^{\phi_r} = P_X(-t)$. This formula was established for classifying spaces of pure braids by Kohno, and then for complements of arbitrary fiber-type arrangements by Falk and Randell. The LCS formula was related to Koszul duality in [12], and extended to formal spaces X with Koszul cohomology algebra in [10]. Our next result gives an LCS-type formula for the rational homotopy groups of a rescaling of X (under no formality assumptions).

THEOREM 2.3. – *Let Y be a simply-connected CW-space of finite type. Assume $H^*(Y, \mathbb{Q})$ is the k-rescaling of a Koszul algebra. Set $\Phi_r := \mathrm{rank}\, \pi_r(\Omega Y)$. Then $\Phi_r = 0$, if r is not a multiple of $2k$, and the following* homotopy LCS formula *holds*:

$$\prod_{i \geqslant 1} \left(1 - t^{(2k+1)i}\right)^{\Phi_{2ki}} = P_Y(-t). \tag{4}$$

Consequently, $\Omega Y \simeq_{\mathbb{Q}} \prod_{i \geqslant 1}^{w} K(\mathbb{Q}, 2ki)^{\Phi_{2ki}}$. If $H^*(Y, \mathbb{Q}) = H^*(X, \mathbb{Q})[k]$, and $H^*(X, \mathbb{Q})$ is a Koszul algebra, it follows that the rational homotopy type of ΩY is determined by the Poincaré polynomial of X. In particular, the Poincaré series of ΩY is given by $P_{\Omega Y}(t) = P_X(-t^{2k})^{-1}$. In fact, by Milnor–Moore [7], $H_*(\Omega Y, \mathbb{Q}) \cong U(\mathrm{gr}_*(\pi_1 X) \otimes \mathbb{Q}[k])$, as Hopf algebras.

We illustrate these results with some simple examples. In each case, X is a formal space, with $H^*(X, \mathbb{Q})$ a Koszul algebra, and Y is the (unique up to $\mathbb{Q}$-equivalence) formal k-rescaling of X.

- $X = S^1$, $Y = S^{2k+1}$. We have $\pi_1 X = \mathbb{Z}$, and so $\pi_*(\Omega Y) \otimes \mathbb{Q} = \mathbb{L}^*(x)$, the free Lie algebra on a generator x in degree $2k$. Thus, $\Omega S^{2k+1} \simeq_{\mathbb{Q}} K(\mathbb{Q}, 2k)$, a result that goes back to Serre's thesis.
- $X = \bigvee^n S^1$, $Y = \bigvee^n S^{2k+1}$. The associated graded of $\pi_1 X$ was computed by Magnus. We obtain: $\pi_*(\Omega Y) \otimes \mathbb{Q} = \mathbb{L}^*(x_1, \ldots, x_n)$. Hence, $\Phi_r = 0$ if $2k \nmid r$, and $\prod_{i \geqslant 1}(1 - t^{(2k+1)i})^{\Phi_{2ki}} = 1 - nt^{2k+1}$. Thus, $P_{\Omega Y}(t) = (1 - nt^{2k})^{-1}$, a result that goes back to Bott and Samelson.
- $X = \#^g S^1 \times S^1$, $Y = \#^g S^{2k+1} \times S^{2k+1}$. The associated graded of $\pi_1 X$ was computed by Labute. We obtain: $\pi_*(\Omega Y) \otimes \mathbb{Q} = \mathbb{L}^*(x_1, \ldots, x_{2g})/([x_1, x_2] + \cdots + [x_{2g-1}, x_{2g}] = 0)$. Hence, $\Phi_r = 0$ if $2k \nmid r$, and $\prod_{i \geqslant 1}(1 - t^{(2k+1)i})^{\Phi_{2ki}} = 1 - 2gt^{2k+1} + t^{4k+2}$. Thus, $P_{\Omega Y}(t) = (1 - 2gt^{2k} + t^{4k})^{-1}$.

3. Malcev completions and Milnor–Moore groups

Let K be a connected, finite-type CW-complex K, with base-point $\star$. Fix an increasing, exhaustive filtration of K by connected, finite subcomplexes, $\{K_r\}_{r \geqslant 0}$, starting with $K_0 = \star$. Let Y be a based, simply-connected CW-space of finite type, and denote by $[K, \Omega Y]$ the group (under composition of loops) of based homotopy classes of based maps. Since K_r is a finite complex, $[K_r, \Omega Y]$ is a finitely-generated nilpotent group. Define the completion $[K, \Omega Y]^\frown := \varprojlim_r ([K_{r-1}, \Omega Y] \otimes \mathbb{Q})$, and endow it with the inverse limit filtration, $F_r[K, \Omega Y]^\frown := \ker([K, \Omega Y]^\frown \to [K_{r-1}, \Omega Y] \otimes \mathbb{Q})$.

For example, $K = \Omega S^m$ ($m \geqslant 2$) has a cell decomposition with one cell of dimension $r(m-1)$, for each $r \geqslant 0$. Setting K_r equal to the $r(m-1)$-th skeleton, we obtain the filtered group $[\Omega S^m, \Omega Y]^\frown$.

Now let G be an arbitrary finitely-generated group. Then G has a *Malcev completion*, defined as $G \otimes \mathbb{Q} := \varprojlim_r ((G/\Gamma_r G) \otimes \mathbb{Q})$. This group comes equipped with the inverse limit filtration; see [11].

The next theorem lifts the Rescaling Formula (3) from the level of associated graded Lie algebras to the level of filtered groups.

THEOREM 3.1. – *Let Y be a k-rescaling of a space X. Assume that $H^*(X,\mathbb{Q})$ is a Koszul algebra. Then, X is formal if and only if the following* Malcev Formula *holds*:

$$\left[\Omega S^{2k+1},\Omega Y\right]^{\frown}\cong\pi_1 X\otimes\mathbb{Q},\quad \textit{as filtered groups}. \tag{5}$$

Let us sketch the proof of the forward implication. To each complete Lie algebra L, one associates, in a functorial way, a filtered group, called the *exponential group* of L. The underlying set of $\exp(L)$ is just L, while the group law is given by the Campbell–Hausdorff formula; see [11]. We then have:

$$\begin{aligned}\left[\Omega S^{2k+1},\Omega Y\right]^{\frown} &\cong \exp\left(\mathrm{Hom}(H_{>0}\left(\Omega S^{2k+1},\mathbb{Q}\right),\pi_*(\Omega Y)\otimes\mathbb{Q})\right)\\ &\cong \exp\left(\pi_*(\Omega Y)\otimes\mathbb{Q}\{2k+1\}^{\frown}\right)\cong\exp\left(\mathrm{gr}_*(\pi_1 X)\otimes\mathbb{Q}^{\frown}\right)\cong\pi_1 X\otimes\mathbb{Q}.\end{aligned}$$

The key is the first isomorphism, which follows from a theorem of H. Baues [1]. The second isomorphism requires a “rebracketing” of the homotopy Lie algebra. The third one is provided by Theorem 2.1, while the last one uses the formality of X (*see* [14,6]).

Consider now the Milnor–Moore group of degree 0 coalgebra maps from $H_*(K,\mathbb{Q})$ to $H_*(\Omega Y,\mathbb{Q})$, as defined in [7]. There is a natural filtration on $\mathrm{Hom}^{\mathrm{coalg}}(H_*(K,\mathbb{Q}),H_*(\Omega Y,\mathbb{Q}))$, with r-th term equal to the kernel of the map induced by the inclusion $K_{r-1}\to K$; see [3]. Using results of Hilton–Mislin–Roitberg and Scheerer, we show that $\mathrm{Hom}^{\mathrm{coalg}}(H_*(K,\mathbb{Q}),H_*(\Omega Y,\mathbb{Q}))\cong[K,\Omega Y]^{\frown}$. Combined with Theorem 3.1, this proves the following theorem.

THEOREM 3.2. – *Let Y be a k-rescaling of a formal space X. If $H^*(X,\mathbb{Q})$ is a Koszul algebra, then*

$$\mathrm{Hom}^{\mathrm{coalg}}\left(H_*\left(\Omega S^{2k+1},\mathbb{Q}\right),H_*(\Omega Y,\mathbb{Q})\right)\cong\pi_1 X\otimes\mathbb{Q},\quad \textit{as filtered groups}. \tag{6}$$

As noted by Cohen and Gitler in [3], the filtered group $[\Omega S^2,\Omega Y]$ is a particularly interesting object. As a set, it equals $\prod_{r\geqslant 1}\pi_r(\Omega Y)$, thus reassembling all the homotopy groups of Y into a single group, called the “group of homotopy groups” of Y. In this context, we obtain a result similar to Theorem 3.2, with ΩS^{2k+1} replaced by ΩS^2. In the case when X is the configuration space of ℓ distinct points in $\mathbb{C}$, and Y is the configuration space of ℓ distinct points in $\mathbb{C}^{k+1}$, this result answers a question posed by Cohen and Gitler. In the case when $X=\bigvee^n S^1$ and $Y=\bigvee^n S^{2k+1}$, we recover a result of Sato (*see* [4]).

4. Rescaling hyperplane arrangements

Let $\mathcal{A}=\{H_1,\dots,H_n\}$ be an arrangement of hyperplanes in $\mathbb{C}^\ell$, with complement $X=M(\mathcal{A})$. For each $k\geqslant 1$, let $\mathcal{A}^k:=\{H_1^{\times k},\dots,H_n^{\times k}\}$ be the corresponding *redundant* arrangement of codimension k subspaces in $\mathbb{C}^{k\ell}$. Then, as shown by Cohen, Cohen and Xicoténcatl [2], the complement $Y=M(\mathcal{A}^{k+1})$ is an integral k-rescaling of X. By work of Brieskorn and Yuzvinsky, respectively, it is known that both X and Y are formal spaces.

By Theorems 2.1 and 2.2, the Rescaling Formula (3) holds precisely for the class of arrangements for which $H^*(X,\mathbb{Q})$ is a Koszul algebra. In this case, Y is coformal, and the Malcev Formula (5) also applies.

Presently, the only arrangements which are known to be Koszul are the fiber-type (or, supersolvable) arrangements; see [12]. For such arrangements, the Rescaling Formula was first established in [2], as a generalization of previous results of Fadell–Husseini and Cohen–Gitler on configuration spaces.

The Poincaré polynomial of the complement of a fiber-type arrangement $\mathcal{A}$, with exponents $d_1,\dots,d_\ell$, factors as $P_X(t)=\prod_{j=1}^{\ell}(1+d_jt)$. From Theorem 2.3, we see that $\Phi_r=\mathrm{rank}\,\pi_r(\Omega Y)$ vanishes if $2k\nmid r$, and $\prod_{i\geqslant 1}(1-t^{(2k+1)i})^{\Phi_{2ki}}=\prod_{j=1}^{\ell}(1-d_jt^{2k+1})$. As a consequence, the rational homotopy type of ΩY is determined solely by the exponents of $\mathcal{A}$. In particular, $P_{\Omega Y}(t)=\prod_{j=1}^{\ell}(1-d_jt^{2k})^{-1}$.

If $\mathcal{A}$ is an affine, generic arrangement of n hyperplanes in $\mathbb{C}^{n-1}$ ($n>2$), then the Rescaling Formula fails for $X=M(\mathcal{A})$, due to the non-coformality of $Y=M(\mathcal{A}^{k+1})$. The absence of coformality is detected by higher-order Whitehead products, which also account for the deviation from equality in (3) and (4). For example, $\Phi_{(2k+1)n-2}=1$, whereas $\mathrm{gr}_{>1}(\pi_1 X)=0$.

5. Rescaling links in S^3

Let $K=(K_1,\ldots,K_n)$ be a link of oriented circles in S^3. For each $k\geqslant 1$, we define the *k-rescaling* $K^{\circledast k}$ to be the link of $(2k+1)$-spheres in S^{4k+3} obtained by taking the iterated join (in the sense of Koschorke and Rolfsen [5]) of the link K with k copies of the n-component Hopf link.

Let X and Y be the complements of K and $K^{\circledast k}$. Clearly, $\pi_1(Y)=0$. Interpreting cup products in X and Y in terms of linking numbers, we show that Y is an integral k-rescaling of X. Since $H^{>2}(X,\mathbb{Z})=0$, this rescaling is unique (up to $\mathbb{Q}$-equivalence), and so Y is a formal space.

Associated to K there is a linking graph, G_K, with vertices corresponding to the components K_i, and edges connecting pairs of distinct vertices for which $\operatorname{lk}(K_i,K_j)\neq 0$. It is known that G_K is connected if and only if $H^*(X,\mathbb{Q})$ is Koszul; see [6]. Examples of links with complete (hence, connected) linking graphs include algebraic links and singularity links of central arrangements of transverse planes in $\mathbb{R}^4$.

The Rescaling Formula (3) holds for X and Y if and only if G_K is connected. In that case, Y is also coformal. Its homotopy Lie algebra is a semidirect product of free Lie algebras generated in degree $2k$, with non-zero ranks given by $\prod_{i\geqslant 1}(1-t^{(2k+1)i})^{\Phi_{2ki}}=(1-t^{2k+1})(1-(n-1)t^{2k+1})$.

On the other hand, the Malcev Formula (5) may fail (due to the non-formality of X), even when the Rescaling Formula does hold. To illustrate this phenomenon, we use the *Campbell–Hausdorff invariants* of links, introduced in [8]. If K_0 and K are two links with the same connected weighted linking graph, and if both link complements are formal, we show that $p^r(K_0)=p^r(K)$, for all $r\geqslant 1$.

Now take K_0 to be the n-component Hopf link ($n\geqslant 4$), and add the Borromean braid on three of its strands to get K. Then K_0 and K have the same weighted linking graph (the complete graph on n vertices, with all linking numbers equal to 1), but $p^3(K_0)\neq p^3(K)$. Since X_0 is obviously formal, X must be non-formal. Hence, if Y is the complement of the k-rescaling of K, then $\pi_*(\Omega Y)\otimes\mathbb{Q}\cong \operatorname{gr}_*(\pi_1 X)\otimes\mathbb{Q}[k]$, as graded Lie algebras, but $[\Omega S^{2k+1},\Omega Y]^{\frown}\ncong \pi_1 X\otimes\mathbb{Q}$, as filtered groups.

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