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Théorie des groupes/*Group Theory*

The tree lattice existence theorems

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Abstract Let X be a locally finite tree, and let $G = \mathrm{Aut}(X)$. Then G is a locally compact group. In analogy with Lie groups, Bass and Lubotzky conjectured that G contains *lattices*, that is, discrete subgroups whose quotient carries a finite invariant measure. Bass and Kulkarni showed that G contains uniform lattices if and only if G is unimodular and $G\backslash X$ is finite. We describe the necessary and sufficient conditions for G to contain lattices, both uniform and non-uniform, answering the Bass–Lubotzky conjectures in full. ***To cite this article: L. Carbone, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 223–228.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Les théorèmes d'existence de réseaux associés aux arbres

Résumé Soit X un arbre localement fini, et soit $G = \mathrm{Aut}(X)$. Alors G est un groupe localement compact. Par analogie avec les groupes de Lie, Bass et Lubotzky ont conjecturé que G contient des *réseaux*, c'est-à-dire des sous-groupes discrets dont le quotient porte une mesure invariante finie. Bass et Kulkarni ont montré que G contient des réseaux uniformes si et seulement si G est unimodulaire et $G\backslash X$ est fini. Nous décrivons les conditions nécessaires et suffisantes pour que G contienne des réseaux, non seulement uniformes mais aussi non-uniformes, prouvant ainsi complètement les conjectures de Bass et Lubotzky. ***Pour citer cet article : L. Carbone, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 223–228.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Soit X un arbre localement fini, et soit $G = \mathrm{Aut}(X)$. Alors G est un groupe localement compact avec stabilisateurs de sommets compacts et ouverts. Soit μ une mesure de Haar invariante à gauche sur G. Un sous-groupe discret Γ de G est appelé un G-*réseau* si $\mu(\Gamma\backslash G)$ est finie ; on dira de plus que Γ est un G-*réseau uniforme* (*ou cocompact*) si $\Gamma\backslash G$ est compact, et qu'il est un G-*réseau non-uniforme* dans le cas contraire.

Par analogie avec les groupes de Lie, Bass et Lubotzky ont conjecturé que le groupe $G = \mathrm{Aut}(X)$ contient des G-réseaux. Bass et Kulkarni on montré [3] que G contient des réseaux uniformes si et seulement si G est unimodulaire et $G\backslash X$ est fini. Dans cette Note, nous décrivons les conditions nécessaires et suffisantes pour que G contienne des réseaux, non seulement uniformes mais aussi non-uniformes, prouvant ainsi

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complètement les conjectures de Bass et Lubotzky. Les détails de ces résultats se trouvent dans les travaux [2,6–8], et [9].

Un sous-groupe $\Gamma \leqslant G$ est *discret* si et seulement si Γ_x est un groupe fini pour chaque $x \in VX$. Pour $\Gamma \leqslant G$ discret, on définit

$$\mathrm{Vol}(\Gamma\backslash\backslash X) := \sum_{x \in V(\Gamma\backslash X)} \frac{1}{|\Gamma_x|},$$

et on appelle Γ un *X-réseau* si $\mathrm{Vol}(\Gamma\backslash\backslash X)$ est fini ; on dira de plus que Γ est un *X-réseau uniforme* si $\Gamma\backslash X$ est un graphe fini, et qu'il est un *X-réseau non-uniforme* dans le cas contraire.

D'après un résultat de Bass et Lubotzky (Théorème (1.1), [4]), on peut construire un G-réseau via la construction d'un X-réseau. Le théorème d'existence uniforme de Bass et Kulkarni s'écrit :

THÉORÈME 0.1 ([3]). – *Soit X un arbre localement fini et soit $G = \mathrm{Aut}(X)$. Les conditions suivantes sont équivalentes* :
(a) *G contient un X-réseau uniforme Γ, qui est aussi un G-réseau uniforme* ;
(b) *G contient un X-réseau uniforme Φ tel que $\Phi\backslash X = G\backslash X$* ;
(c) *G est unimodulaire et $G\backslash X$ est fini* ;
(d) *X est le revêtement universel d'un graphe fini connexe.*
Sous ces conditions, X est appelé « arbre uniforme ».

Lorsque G est unimodulaire, $\mu(G_x)$ est constant sur les orbites de G, et on peut définir ([4], (1.5)) :

$$\mu(G\backslash\backslash X) := \sum_{x \in V(G\backslash X)} \frac{1}{\mu(G_x)}.$$

Dans [2], nous avons prouvé le théorème suivant :

THÉORÈME 0.2 ([2]). – *Soit X un arbre localement fini, soit $G = \mathrm{Aut}(X)$, et soit μ une mesure de Haar invariante à gauche sur G. Supposons que G est unimodulaire, que $\mu(G\backslash\backslash X) < \infty$, et que $G\backslash X$ est infini. Alors G contient un X-réseau Γ (nécessairement non-uniforme).*

Les Théorèmes 0.1 et 0.2 entraînent le résultat suivant, qui apparaît sous forme de conjecture dans une version antérieure de [4] :

THÉORÈME 0.3 ([2]). – *Soit X un arbre localement fini, soit $G = \mathrm{Aut}(X)$, et soit μ une mesure de Haar invariante à gauche sur G. Les conditions suivantes sont équivalentes* :
(a) *G contient un X-réseau Γ* ;
(b) *G est unimodulaire et $\mu(G\backslash\backslash X) < \infty$.*

Concernant l'existence de G-réseaux non-uniformes, on a :

THÉORÈME 0.4 ([9]). – *Soit X un arbre localement fini avec plus d'une extrémité, et soit $G = \mathrm{Aut}(X)$. Les conditions suivantes sont équivalentes* :
(a) *G contient un X-réseau non-uniforme* ;
(b) *G contient un G-réseau non-uniforme et $\mu(G\backslash\backslash X) < \infty$.*

Si X est uniforme, alors (a) $\Rightarrow$ (b) est immédiat, et le problème de l'existence d'un (X- ou G-) réseau non-uniforme est résolu dans [6] et [7]. Si X n'a qu'une extrémité, on a :

THÉORÈME 0.5 ([8]). – *Soit X un arbre localement fini et soit $G = \mathrm{Aut}(X)$. Si X a une extrémité unique, et si G contient un X-réseau non-uniforme, alors G contient un G-réseau non-uniforme si et seulement si les indices sur les arêtes du quotient de X ne sont pas bornés le long de tout chemin dirigé vers son extrémité.*

Par analogie avec le théorème classique de Borel établissant la coexistence de réseaux uniformes et non-uniformes dans les groupes de Lie semisimples non-compacts connexes, et avec le théorème de Lubotzky concernant la coexistence de réseaux uniformes et non-uniformes dans les groupes algébriques simples de rang relatif 1 sur des corps locaux non-archimédiens [13], Bass et Lubotzky ont conjecturé (dans une version antérieure de [4]) que, moyennant des hypothèses naturelles, $G = \mathrm{Aut}(X)$ contient des réseaux *à la fois* uniformes et non-uniformes.

Afin de formuler notre résultat, on dit que X est *rigide* si G est discret, et qu'il est *minimal* si G agit minimalement sur X, c'est-à-dire s'il n'existe pas de sous-arbre propre G-invariant. Si X est uniforme, alors il y a toujours un sous-arbre G-invariant minimal unique $X_0 \subseteq X$ ([4] (5.7), (5.11), (9.7)). On dit que X est *virtuellement rigide* si X_0 est rigide. D'après un résultat de Bass et Tits [5], si X est uniforme et rigide, alors tous les X-réseaux doivent être uniformes. Il s'ensuit [4] que si X est uniforme et virtuellement rigide, tous les X-réseaux sont uniformes. Réciproquement, on a :

THÉORÈME 0.6 ([6,7]). – *Si X n'est pas virtuellement rigide et $G = \mathrm{Aut}(X)$ contient un X-réseau uniforme, alors G contient un X-réseau non-uniforme Γ, qui est aussi* (*nécessairement*) *un G-réseau non-uniforme.*

Dans [6], nous avons prouvé le Théorème 0.6 pour les actions minimales, en supposant aussi les critères (nécessaires) de Bass et Tits pour que G ne soit pas discret [5]. Dans [7], nous avons prouvé le Théorème 0.6 dans le cas où la restriction de G au sous-arbre G-invariant minimal unique $X_0 \subseteq X$ n'est pas discrète.

Let X be a locally finite tree, and let $G = \mathrm{Aut}(X)$. The stabilizers of finite sets of vertices form a fundamental system of profinite neighbourhoods of the identity in G. Thus G is a locally compact totally disconnected group with compact open vertex stabilizers, G_x for $x \in VX$.

Let μ be a left invariant Haar measure on G. We call a discrete subgroup Γ of G a *G-lattice* if $\mu(\Gamma\backslash G)$ is finite, and a *uniform* (*or cocompact*) *G-lattice* if $\Gamma\backslash G$ is compact, a *non-uniform G-lattice* otherwise.

Hyman Bass and Alex Lubotzky initiated a program to study the group $G = \mathrm{Aut}(X)$ in analogy with non-compact simple Lie groups (*see* [4] and [14]). This program is motivated by the case of a simple algebraic K-group H, of K-rank 1, over a non-archimedan local field K, with finite residue field $\mathbb{F}_q$. The group $H \leqslant \mathrm{Aut}(X)$ acts on its Bruhat–Tits tree X; for example, if $H = \mathrm{PSL}_2(K)$ then X is the homogeneous tree X_{q+1}.

In analogy with the Lie group case, Bass and Lubotzky conjectured that the group $G = \mathrm{Aut}(X)$ contains G-lattices. Bass and Kulkarni showed [3] that G contains uniform lattices if and only if G is unimodular and $G\backslash X$ is finite. Here describe the necessary and sufficient conditions for G to contain lattices, both uniform and non-uniform, answering the Bass–Lubotzky conjectures in full. The details of these results are contained within the works [2,6–8], and [9]. Our method is constructive. In each case we show that lattices exist by constructing them.

A natural approach to the constructive problem of producing a G-lattice is suggested by the fundamental theory of Bass–Serre [1,16] which states that any action without inversions of a group on a tree is encoded in a 'quotient graph of groups'. To construct a G-lattice we may hence construct instead the appropriate graph of groups. By passing to a subgroup G^+ of G of index two (or to a barycentric subdivision of X) if necessary, we may assume that all groups $\Gamma \leqslant G^+$ act on X without inversions.

In order to determine the structure of the graph of groups of a G-lattice we appeal to the topology on G which gives the following criterion for discreteness. A subgroup $\Gamma \leqslant G$ is *discrete* if and only if Γ_x is a finite group for each $x \in VX$. For discrete $\Gamma \leqslant G$ we define

$$\mathrm{Vol}(\Gamma\backslash\backslash X) := \sum_{x \in V(\Gamma\backslash X)} \frac{1}{|\Gamma_x|},$$

and call Γ an *X-lattice* if $\mathrm{Vol}(\Gamma\backslash\backslash X)$ is finite, a *uniform X-lattice* if $\Gamma\backslash X$ is a finite graph, a *non-uniform X-lattice* otherwise.

For discrete $\Gamma \leqslant G = \mathrm{Aut}(X)$, the relationship between covolumes $\mu(\Gamma\backslash G)$ and $\mathrm{Vol}(\Gamma\backslash\backslash X)$ is given by the following (Theorem (1.1)). When G is unimodular, $\mu(G_x)$ is constant on G-orbits, so we can define ([4], (1.5)):

$$\mu(G\backslash\backslash X) := \sum_{x\in V(G\backslash X)} \frac{1}{\mu(G_x)}.$$

THEOREM 0.1 ([4], (1.6)). – *Let X be a locally finite tree. For a discrete subgroup $\Gamma \leqslant G = \mathrm{Aut}(X)$, the following conditions are equivalent*:
(a) *Γ is an X-lattice, that is,* $\mathrm{Vol}(\Gamma\backslash\backslash X) < \infty$.
(b) *Γ is a G-lattice* (*hence G is unimodular*), *and* $\mu(G\backslash\backslash X) < \infty$.
In this case:

$$\mathrm{Vol}(\Gamma\backslash\backslash X) = \mu(\Gamma\backslash G)\cdot\mu(G\backslash\backslash X).$$

Applying Theorem 0.1, we construct a G-lattice by constructing instead an X-lattice.

We have the following result, originally conjectured in an earlier version of [4]:

THEOREM 0.2 ([2]). – *Let X be a locally finite tree, $G = \mathrm{Aut}(X)$, and let μ be a left Haar measure on G. Equivalent conditions*:
(a) *G contains an X-lattice Γ.*
(b) *G is unimodular and $\mu(G\backslash\backslash X) < \infty$.*

The implication (a) $\Rightarrow$ (b) of Theorem 0.2 follows from Theorem 0.1.

When $G\backslash X$ is finite, we have:

THEOREM 0.3 ([3]). – *Let X be a locally finite tree and let $G = \mathrm{Aut}(X)$. The following conditions are equivalent*:
(a) *G contains a uniform X-lattice Γ, which is also a uniform G-lattice.*
(b) *G contains a uniform X-lattice Φ such that $\Phi\backslash X = G\backslash X$.*
(c) *G is unimodular and $G\backslash X$ is finite.*
(d) *X is the universal cover of a finite connected graph.*
Under these conditions, X is called a 'uniform tree'.

The following result, together with Theorem 0.3 gives Theorem 0.2.

THEOREM 0.4 ([2]). – *Let X be a locally finite tree, let $G = \mathrm{Aut}(X)$, and let μ be a left Haar measure on G. Assume that G is unimodular, $\mu(G\backslash\backslash X) < \infty$, and $G\backslash X$ is infinite. Then G contains a* (*necessarily non-uniform*) *X-lattice Γ.*

The assumptions that $G\backslash X$ is infinite and $\mu(G\backslash\backslash X) < \infty$ imply that G itself is not discrete, which is a necessary condition for the existence of a non-uniform X-lattice [5]. If $G\backslash X$ is instead finite, it is necessary to assume that G is not discrete in order to construct a non-uniform X-lattice (*see* Theorem 0.8).

The X-lattice Γ constructed in Theorem 0.4 turns out to be a *uniform G*-lattice. Let Γ be a non-uniform X-lattice. By Theorem 0.1, Γ is a G-lattice and the diagram of natural projections

$$\begin{array}{ccc} & X & \\ {\scriptstyle p_\Gamma}\swarrow & & \searrow{\scriptstyle p_G} \\ \Gamma\backslash X & \xrightarrow{\;p\;} & G\backslash X \end{array}$$

commutes. To determine if Γ is uniform or non-uniform as a G-lattice, we use the following:

LEMMA 0.5 ([4], (1.5.8)). – *Let $x \in VX$. The following conditions are equivalent*:

(a) Γ *is a uniform* G*-lattice.*
(b) *Some fiber* $p^{-1}(p_G(x)) \cong \Gamma\backslash G/G_x$ *is finite.*
(c) *Every fiber of* p *is finite.*

It follows that if $G\backslash X$ is finite, then Γ is a uniform (respectively non-uniform) X-lattice if and only if Γ is a uniform (respectively non-uniform) G-lattice. If we assume that $G\backslash X$ is infinite, to construct a non-uniform G-lattice our task is to construct a discrete group Γ with $\Gamma\backslash X$ infinite, $\mathrm{Vol}(\Gamma\backslash\backslash X) < \infty$, and some (hence every) fiber of the projection p infinite.

For the existence of non-uniform G-lattices, we have:

THEOREM 0.6 ([9]). – *Let* X *be a locally finite tree with more than one end, and let* $G = \mathrm{Aut}(X)$*. The following conditions are equivalent*:
(a) G *contains a non-uniform* X*-lattice.*
(b) G *contains a non-uniform* G*-lattice and* $\mu(G\backslash\backslash X) < \infty$*.*

If X is uniform, then (a) $\Rightarrow$ (b) is automatic, and question of the existence of a non-uniform (X- or G-) lattice is answered in [6] and [7]. If X has only one end, we have:

THEOREM 0.7 ([8]). – *Let* X *be a locally finite tree and let* $G = \mathrm{Aut}(X)$*. If* X *has a unique end, and if* G *contains a non-uniform* X*-lattice, then* G *contains a non-uniform* G*-lattice if and only if every path directed towards the end of the edge-indexed quotient of* X *has unbounded index.*

Suppose now that $G = \mathrm{Aut}(X)$ is *compact*. Then any lattice (or even discrete) subgroup is finite. Hence G will not contain any X-lattices unless X itself, and so also G, is finite. In this case G is then itself a uniform X-lattice, so it cannot contain a non-uniform X-lattice.

In analogy with Borel's classical theorem establishing the co-existence of uniform and non-uniform lattices in connected non-compact semisimple Lie groups, and Lubotzky's theorem concerning the co-existence of uniform and non-uniform lattices in simple algebraic groups of relative rank 1 over non-archimedean local fields [13], Bass and Lubotzky conjectured (in an earlier version of [4]) that under some natural assumptions $G = \mathrm{Aut}(X)$ contains *both* uniform and non-uniform lattices. We have obtained a positive answer to this conjecture ([6] and [7]).

In order to state our results, we call X *rigid* if G is discrete, and we call X *minimal* if G acts minimally on X, that is, there is no proper G-invariant subtree. If X is uniform then there is always a unique minimal G-invariant subtree $X_0 \subseteq X$ ([4] (5.7), (5.11), (9.7)). We call X *virtually rigid* if X_0 is rigid. By a result of Bass–Tits [5], if X is uniform and rigid then all X-lattices must be uniform. It follows [4] that if X is uniform and virtually rigid, all X-lattices are uniform. Conversely we have:

THEOREM 0.8 ([6,7]). – *If* X *is not virtually rigid and* $G = \mathrm{Aut}(X)$ *contains a uniform* X*-lattice, then* G *contains a non-uniform* X*-lattice* Γ*, which is also* (*necessarily*) *a non-uniform* G*-lattice.*

In [6], we proved Theorem 0.8 for minimal actions assuming also the (necessary) Bass–Tits criterion for non-discreteness of G [5]. In [7] we proved Theorem 0.8 in the case that the restriction of G to the unique minimal G-invariant subtree $X_0 \subseteq X$ is not discrete.

The following theorem demonstrates that any positive real number can occur as the covolume of a non-uniform lattice on a uniform tree.

THEOREM 0.9 ([15]). – *Let* X *be a uniform tree which is not virtually rigid and let* $G = \mathrm{Aut}(X)$*. If* $v \in \mathbb{R}_{>0}$ *then there exists a non-uniform* X*-lattice* Γ *such that* $\mathrm{Vol}(\Gamma\backslash\backslash X) = v$*.*

Since the covolume of a lattice is constant on conjugacy classes, we deduce that the number of conjugacy classes of non-uniform X-lattices on uniform trees is uncountable.

We have also strengthened the existence theorems for non-uniform X-lattices to include infinite towers of X-lattices:

THEOREM 0.10 ([10,11]). – *Let X be a locally finite tree. If X has more than one end, and $G = \mathrm{Aut}(X)$ contains a non-uniform X-lattice Γ then G contains an infinite ascending chain*

$$\Gamma_1 < \Gamma_2 < \Gamma_3 < \cdots$$

of non-uniform X-lattices. Hence $\mathrm{Vol}(\Gamma_i \backslash\backslash X) \to 0$ *as* $i \to \infty$.

The Kazhdan–Margulis property for lattices in Lie groups [12] states that the covolume of a lattice is bounded away from zero. Hence the existence of infinite towers of X-lattices in $G = \mathrm{Aut}(X)$ shows that the Kazhdan–Margulis property is violated for X-lattices.

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References

[1] H. Bass, Covering theory for graphs of groups, J. Pure Appl. Algebra 89 (1993) 3–47.
[2] H. Bass, L. Carbone, G. Rosenberg, The existence theorem for tree lattices, Appendix [BCR], in: H. Bass, A. Lubotzky, Tree Lattices, Progr. Math., Vol. 176, Birkhäuser, Boston, 2000.
[3] H. Bass, R. Kulkarni, Uniform tree lattices, J. Amer. Math. Soc. 3 (4) (1990).
[4] H. Bass, A. Lubotzky, Tree Lattices, Progr. Math., Vol. 176, Birkhäuser, Boston, 2000.
[5] H. Bass, J. Tits, A discreteness criterion for certain tree automorphism groups, Appendix [BT], in: H. Bass, A. Lubotzky, Tree Lattices, Progr. Math., Vol. 176, Birkhäuser, Boston, 2000.
[6] L. Carbone, Non-uniform lattices on uniform trees, Mem. Amer. Math. Soc. 152 (724) (2001).
[7] L. Carbone, Non-minimal tree actions and the existence of non-uniform tree lattices, Preprint, 2002.
[8] L. Carbone, D. Clark, Lattices on parabolic trees, Comm. Algebra 30 (4) (2002).
[9] L. Carbone, G. Rosenberg, Lattices on non-uniform trees, Geom. Dedicate (2002), to appear.
[10] L. Carbone, G. Rosenberg, Infinite towers of tree lattices, Math. Res. Lett. 8 (2001) 1–10.
[11] L. Carbone, G. Rosenberg, Infinite towers of non-uniform tree lattices (2002), in preparation.
[12] D. Kazhdan, G. Margulis, A proof of Selberg's hypothesis, Mat. Sb. (N.S.) 75 (117) (1968) 163–168 (in Russian).
[13] A. Lubotzky, Lattices in rank one Lie groups over local fields, Geom. Funct. Anal. 1 (4) (1991) 405–431.
[14] A. Lubotzky, Tree lattices and lattices in Lie groups, in: A. Duncan, N. Gilbert, J. Howie (Eds.), Combinatorial and Geometric Group Theory, LMS Lecture Note Series, Vol. 204, Cambridge University Press, 1995, pp. 217–232.
[15] G. Rosenberg, Towers and covolumes of tree lattices, Ph.D. Thesis, Columbia University, 2000.
[16] J.-P. Serre, Trees, Springer-Verlag, Berlin, 1980. Translated from the French by J. Stilwell.