**C. R. Acad. Sci. Paris, Ser. I 335 (2002) 355–358**

**Géométrie algébrique/*Algebraic Geometry***

# Deformations of locally complete intersections

**Catriona Maclean**

**Institut de mathématiques de Jussieu, Université Paris 6, 175, rue de Chevaleret, 75013, Paris, France**

**Received 8 November 2001; accepted after revision 1 July 2002**

**Note presented by Jean-Pierre Serre.**

---

**Abstract** Given a projective l.c.i. scheme, $X \subset \mathbb{P}^N$, we show that $X$ has a smooth formal neighbourhood in which $X$ is globally a complete intersection; that is, $X$ is the intersection of $\operatorname{codim}(X)$ hypersurfaces. ***To cite this article: C. Maclean, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 355–358.***

## Déformations des schémas localement intersections complètes

**Résumé** Soit $X$ un schéma projectif et localement intersection complète. On démontre qu'il existe un voisinage formel, $X_\infty$, de $X$, dans lequel $X$ est une intersection complète globale ; c'est-à-dire que $X$ est l'intersection de $\operatorname{codim}(X)$ hypersurfaces. ***Pour citer cet article : C. Maclean, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 355–358.***

---

## 1. Introduction

If $X \subset \mathbb{P}^N(k)$ ($k$ being any field) is a projective local complete intersection scheme, then $X$ is not necessarily a global complete intersection in projective space – that is, $X$ is not necessarily embedded in $\mathbb{P}^N(k)$ as the vanishing locus of codim $X$ polynomials. It seems natural to ask whether this is true for more general ambient varieties. In particular, given such an $X$, we may wonder whether it can be embedded in some smooth $Y$ as a globally complete intersection; i.e., the intersection of $\operatorname{codim}(X)$ hypersurfaces. The aim of this Note is to answer this question in the affirmative, at least formally, by proving the following result:

THEOREM 1.1. – *Let $X \subset \mathbb{P}^N(k)$ be a projective local complete intersection scheme. Then there exists a smooth formal neighbourhood, $X_\infty$ of $X$, a vector bundle, $V$ on $X_\infty$, and a section $\sigma : X_\infty \to V$, such that*

- *$V$ is a direct sum of line bundles,*
- *$rk\ (V) = \operatorname{codim}_{X_\infty} X$,*
- *$X$ is schematically the zero locus of $\sigma$.*

*Remark*. – I do not know whether or not this scheme $X_\infty$ may be choosen algebrisable.

---

*E-mail address:* maclean@clipper.ens.fr (C. Maclean).

S1631-073X(02)02490-1/FLA

### 1.1. Idea of the proof

We will embed $\mathbb{P}^N$ in a smooth space, $W$. The normal bundle of $\mathbb{P}^N$ in $W$ will be highly negative, and $\mathbb{P}^N$ will be the zero locus of a section of a vector bundle, $V$, which is a direct sum of line bundles. We will consider the spaces $\mathbb{P}^N_n$, which will be cut out in $W$ by $I^n_{\mathbb{P}^N}$, and will recursively construct an l.c.i. scheme $X_n$ in $\mathbb{P}^N_n$ extending $X_{n-1}$. If $V$ is negative enough, the construction of $X_n$ will be unobstructed, and we may therefore continue this construction to $\infty$ to obtain a formal neighbourhood, $X_\infty$ of $X$. We will also be able to impose that $X_\infty$ is smooth. $X_\infty$ will then satisfy all the requirements of the theorem.

## 2. Proof of Theorem 1

Let $I_X$ be the ideal sheaf of $X$ in $\mathbb{P}^N$. Since $X$ is a l.c.i. subscheme of $\mathbb{P}^N$, the co-normal bundle $I_X/I_X^2$ is a locally free sheaf of $\mathcal{O}_X$ modules. We recall Serre's vanishing theorem, which may be found in [1]

PROPOSITION 2.1. – *Let $F$ be a coherent sheaf on $X$, a projective scheme. There exists an $i$ such that, for all $a \geqslant i$ and for all $j \geqslant 1$, we have*:

1. $H^j(X, F(a)) = 0$,
2. *$F(a)$ is generated by its global sections.*

We may therefore, in particular, choose $m$ such that

1. $H^1((I_X/I_X^2)^* \otimes \mathcal{O}_X(m)) = 0$;
2. $(I_X/I_X^2)^* \otimes \mathcal{O}_X(m)$ is generated by its global sections.

We define:

- $l$, the dimension of $H^0((I_X/I_X^2)^* \otimes \mathcal{O}_X(m))$;
- $W$, the total space of the vector bundle $\mathcal{O}_{\mathbb{P}^N}(-m)^{\oplus l}$, in which $\mathbb{P}^N$ is naturally embedded as the zero section;
- $\pi$, the projection $\pi : W \to \mathbb{P}^N$.

We denote by $V$ the bundle $\mathcal{O}_{\mathbb{P}^N}(m)^{\oplus l}$ and by $\mathbb{P}^N_n$ the n-th formal neighbourhood of $\mathbb{P}^N$ in $W$, that is, the subscheme defined by the ideal $I^n_{\mathbb{P}^N}$. Let $c$ be the codimension of $X$ in $\mathbb{P}^N$.

The following proposition allows us to recursively construct neighbourhoods of $X$ in $\mathbb{P}^N_n$ in a compatible way:

PROPOSITION 2.2. – *If $X_n$ is a l.c.i. subscheme of $\mathbb{P}^N_n$, such that $X_n \cap \mathbb{P}^{\mathbb{N}} = X$, and codim $X_n \subset \mathbb{P}^N_n = c$, then there exists $X_{n+1}$, an l.c.i. subscheme of $\mathbb{P}^N_{n+1}$ such that*

- $X_{n+1} \cap \mathbb{P}^N_n = X_n$,
- *codim* $(X_{n+1}) = c$.

*Proof.* – If $U$ is an open subscheme of $\mathbb{P}^N$, then we denote by $U_n$ the open subscheme of $\mathbb{P}^N_n$ whose underlying geometric space is $U$. Let

- $U^i$ be an affine open covers of $\mathbb{P}^N$, such that $X_n \cap U^i_n$ is a complete intersection,
- $f^i_1, \dots, f^i_c \subset H^0(U^i_n, I_{X^n \cap U^i_n})$ a regular sequence for the ideal sheaf of $X_n \cap U^i_n$.

We denote $U^i \cap U^j$ by $U^{i,j}$, $X_n \cap U^i_n$ by $X^i_n$ and $X_n \cap U^{i,j}_n$ by $X^{i,j}_n$.

For every $i$, $d$, we choose $\tilde{f}^i_d \in H^0(U^i_{n+1}, \mathcal{O}_{\mathbb{P}^N_{n+1}})$ such that $\tilde{f}^i_d|_{\mathbb{P}^N_n} = f^i_d$. $\tilde{f}^i_d$ is then a regular sequence in $\mathcal{O}_{U^i_{n+1}}$. We denote by $I^i_{n+1}$ the ideal sheaf of $\mathcal{O}_{U^i_{n+1}}$ generated by the $\tilde{f}^i_d$'s.

$I^i_{n+1}$ defines an l.c.i subscheme of $U^i_{n+1}$, which we denote by $\widetilde{X}^i_{n+1}$. We will show that, after modification of the functions $\tilde{f}^i_d$, $I^i_{n+1}$ will be equal to $I^j_{n+1}$ on $U^{i,j}_{n+1}$ and therefore, the $\widetilde{X}^i_{n+1}$'s may be glued together to form an l.c.i subscheme, $X_{n+1} \subset \mathbb{P}^N_{n+1}$, satisfying the requirements of the proposition.

Consider the following exact sequence of $\mathcal{O}_{\mathbb{P}^N_{n+1}}$ modules:

$$0 \to \mathrm{Sym}^{n+1}(V) \to \mathcal{O}_{\mathbb{P}^N_{n+1}} \to \mathcal{O}_{\mathbb{P}^N_n} \to 0.$$

$\mathrm{Sym}^{n+1}(V)$ is here considered with its $\mathcal{O}_{\mathbb{P}^N}$-module structure. We will now define a Cech cocycle

$$h^{i,j} \in \Gamma\big(\big(I_{X_n}/I_{X_n}^2\big)^* \otimes_{\mathcal{O}_{X_n}} \mathrm{Sym}^{n+1}(V|_X), U_n^{i,j}\big)$$

whose vanishing will be a sufficient condition for $\widetilde{X}_{n+1}^i$ to be compatible.

Given $f \in \Gamma(I_{X_n}/I_{X_n}^2, U_n^{i,j})$, we now construct $h(f) \in \Gamma(\mathrm{Sym}^{n+1}(V|_X), U^{i,j})$. We choose

- $f' \in \Gamma(I_{X_n}, U_n^{i,j})$, extending $f$. This is possible since, $U_n^i$ and $U_n^j$, and therefore $U_n^{i,j}$ are affine.
- $f'^i \in \Gamma(I_{n+1}^i, U_{n+1}^{i,j})$ extending $f'$.
- $f'^j \in \Gamma(I_{n+1}^j, U_{n+1}^{i,j})$ extending $f'$.

Then, $f'^i - f'^j \in \mathrm{Sym}^{n+1}(V)$ and $f'^i - f'^j|_X \in \mathrm{Sym}^{n+1}(V)|_X$.

The difference $f'^i - f'^j|_X$ is independent of the choice of $f'^i$. Indeed, the choice of a different $f'^i$ alters $f'^i - f'^j$ by an element of $I_{n+1}^i \cap \mathrm{Sym}^{n+1}(V)$, which is equal to $\mathrm{Sym}^{n+1}(V) \otimes I_X$, since $f_1^i, \ldots, f_c^i$ is a regular sequence. The same argument show that $f'^i - f'^j|_X$ is independent of the choice of $f'^j$.

Likewise, $f'^i - f'^j|_X$ is independent of $f'$. If $f'' = f' + g_1 g_2$, $g_i \in I_{X_n}$, then we may choose

$$f''^i = f'^i + g_1^i g_2^i, \qquad f''^j = f'^j + g_1^j g_2^i$$

and hence $f''^i - f''^j = (g_1^i - g_1^j) g_2^i + f'^i - f'^j$, whence $(f''^i - f''^j)|_X = (f'^i - f'^j)|_X$.

We may therefore define $h^{i,j}$ by $h^{i,j}(f) = f'^i - f'^j|_X$. We now need the following lemma:

LEMMA 2.3. – *If* $h_{i,j} = 0$, *then* $I_{n+1}^i$ *and* $I_{n+1}^j$ *are compatible on the intersection* $U^{i,j}$.

*Proof.* – If $h^{i,j} = 0$, then, for every $f \in \Gamma(I_{n+1}^i, U^{i,j})$, there exists $g \in \Gamma(I_{n+1}^j, U^{i,j})$, such that

$$(g - f) \in I_X \otimes_{\mathcal{O}_X} \mathrm{Sym}^{n+1}(V).$$

On $U^{i,j}$, we have $I_X \otimes_{\mathcal{O}_{\mathbb{P}^N}} \mathrm{Sym}^{n+1}(V) = I_{n+1}^i \otimes_{\mathcal{O}_{\mathbb{P}_{n+1}^N}} \mathrm{Sym}^{n+1}(V)$ and therefore $(g - f) \in I_{n+1}^j$ whence $f \in I_{n+1}^j$. □

We now finish the proof of the proposition. We alter the regular sequences $\tilde{f}_d^i$ so that $h^{i,j} = 0$. We note that

$$H^1\big(\big(I_{X_n}/I_{X_n}^2\big)^* \otimes_{\mathcal{O}_{X_n}} \mathrm{Sym}^{n+1}(V|_X)\big) = H^1\big(\big(I_X/I_X^2\big)^* \otimes_{\mathcal{O}_X} \mathrm{Sym}^{n+1}(V|_X)\big) = 0,$$

and that, therefore, there exist elements $h_i \in \Gamma((I_{X_n}/I_{X_n}^2)^* \otimes \mathrm{Sym}^n(V|_X), U_i)$, such that

$$h_{i,j} = h_i - h_j.$$

We now choose a new regular sequence $\bar{f}_d^i$, in such a way that $\bar{f}_d^i|_X = \tilde{f}_d^i|_X + h_i(f_d^i)$. These sequences generate new ideal sheaves $\bar{I}_{n+1}^i$. It is immediate from the construction that the associated cocycle $\bar{h}_{i,j}$ is 0, and hence, by the previous lemma, the sheaves $\bar{I}_{n+1}^i$ are compatible on the intersections. Therefore, the $\bar{I}_{n+1}^i$'s glue together to form a global ideal sheaf, $I_{n+1}$. $I_{n+1}$ defines a subscheme of $\mathbb{P}_{n+1}^N$, which we denote by $X_{n+1}$. $X_{n+1}$ is l.c.i, and $\mathrm{codim}(X_{n+1}) = c$ since $\bar{f}_d^i$ is a regular sequence for $X_{n+1} \cap U_{n+1}^i$. $X_{n+1}$ satisfies, therefore, all the requirements of the proposition. □

The formal scheme $\lim n \to \infty X_n$ is then a formal neighbourhood of $X$ in which $X$ is embedded as the zero locus of the tautological section of $\pi^*(\mathcal{O}_X(-m)^{\oplus l}$. It remains only to show that $X_\infty$ is smooth for some choice of $X_n$. The smoothness of $X_\infty$ depends only on the choice of $X_2$. All the results we now quote on Kähler differentials may be found in [1].

Let $\pi$ be the projection $\pi : \mathbb{P}_2^N \to \mathbb{P}^N$. The sheaf of Kähler differentials $\Omega_k^1(\mathbb{P}_2^N) \otimes \mathcal{O}_{\mathbb{P}^N}$ is canonically isomorphic to $\pi^*\Omega_k^1(\mathbb{P}^N) \oplus V$. The universal derivative map: $d : I_{X_2}/I_{X_2}^2 \to \Omega_k^1(\mathbb{P}_2^N) \otimes \mathcal{O}_{X_2}$ is an $\mathcal{O}_{X_2}$ linear map. After tensoring by $\mathcal{O}_X$, we obtain an $\mathcal{O}_X$-linear map:

$$d_{X_2} : I_X/I_X^2 \to \Omega_k^1\big(\mathbb{P}^N\big)|_X \oplus V|_X,$$

$X_\infty$ is smooth at $x$ if $d_{X_2}(x)$ is injective. We now assocaite to any $\phi : I_X/I_X^2 \to V_X$ a candidate space for $X_2$, $X_\phi$, in such a way that $X_\phi$ will be smooth if $\phi(x)$ is injective for all $x$. $X_\phi \subset \mathbb{P}_2^N$ is defined by the formula:

$$f \in I_{X_\phi} \Leftrightarrow f|_{\mathbb{P}^N} \in I_X \quad \text{and} \quad \big(f - \pi^*(f|_{\mathbb{P}^N})\big)|_X = \phi(f).$$

For this $X_\phi$, we have that

$$d_{X_\phi}(f) = \pi^* df + \phi(f),$$

whence $d_{X_\phi}(x)$ is injective for all $x$ if $\phi(x)$ is injective for all $x$. It remains only to find $\phi \in \mathrm{Hom}(I_X/I_X^2, V)$ such that $\phi(x)$ is injective for every $x$. Now, $\mathrm{Hom}(I_X/I_X^2, \mathcal{O}_X(m))$ is globally generated. If $v_1, \ldots, v_l$ is a basis for $H^0(\mathrm{Hom}(I_X/I_X^2, \mathcal{O}_X(m)))$, then

$$\phi = \bigoplus_{b=1}^{l} v_b : I_X/I_X^2 \to V$$

is injective on $I_X/I_X^2(x)$ for every $x$.

*Remark.* – One might wonder whether this work holds for quasi-projective $X$. We have used the fact that $X$ is projective only to invoke Serre's vanishing theorem. Suppose that $X$ is a quasi projective variety – that is, $X$ is the complement in an projective variety, $V$, of an closed subvariety of $V$, $U$. The results of this paper will hold for $X$, provided that:

(A) For any coherent sheaf, $\mathcal{F}$ on $X$, there exists $k$ such that for any $m \geqslant k$ $H^1(\mathcal{F}(m)) = 0$ and $\mathcal{F}(m)$ is generated by its global sections.

If $U$ is of pure codimension 1, then $X$ is affine, and the condition (A) is immediately satisfied. Further, there is an exact sequence:

$$H^1\big(V, \mathcal{F}(m)\big) \to H^1\big(X, \mathcal{F}(m)\big) \to H_U^2\big(V, \mathcal{F}(m)\big),$$

where $H_U^2(V, \mathcal{F}(m))$, the local cohomology of $F$ along $U$, vanishes if every component of $U$ has codimension $\geqslant 3$ (*see* [2] for more details). It follows that the results of this Note are, in fact, also valid for $X = V/U$, where $V$ is projective and $U$ is a closed subvariety containing no codimension 2 component.

**Acknowledgements.** I would like to express my gratitude to my supervisor, Claire Voisin, for her unstinting help and guidance. I would also like to thank Yves Lazlo, who was always ready to answer my questions, and the referee, whose generous comments and help have much improved this work.

## References

[1] R. Hartshorne, Algebraic Geometry, Grad. Texts in Math., Vol. 52, Springer.
[2] A. Grothendieck, S.G.A., Facsimile 1, exposes 1–3, IHES, 1962.