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Chow–Künneth projectors for modular varieties

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Abstract We show the existence of the Chow–Künneth projectors for certain varieties, including Kuga–Shimura varieties of Hilbert modular varieties. The Chow–Künneth projectors of a smooth projective variety are, by definition, mutually orthogonal idempotents of the Chow ring of self-correspondences which give decomposition of the total cohomology of the variety into degree pieces. ***To cite this article: B.B. Gordon et al., C. R. Acad. Sci. Paris, Ser. I 335 (2002) 745–750.***

Projecteurs de Chow–Künneth pour des variétés modulaires

Résumé Nous démontrons l'existence des projecteurs de Chow–Künneth pour certaines variétés, incluant les variétés de Kuga–Shimura des variétés modulaires de Hilbert. Les projecteurs de Chow–Künneth d'une variété lisse projective sont par définition des idempotents orthogonaux de l'anneau de Chow des auto-correspondances qui donnent la décomposition par les degrés de la cohomologie totale de la variété. ***Pour citer cet article : B.B. Gordon et al., C. R. Acad. Sci. Paris, Ser. I 335 (2002) 745–750.***

Version française abrégée

Soit X une variété lisse de dimension d sur un corps k. Les projections de Chow–Künneth de X [13] sont des éléments Π_i, $i = 0, \ldots, 2d$, de l'anneau correspondant $\mathrm{CH}_d(X \times X)$ vérifiant les conditions suivantes :

– Les Π_i sont des projections orthogonales et $\sum_i \Pi_i = \Delta_X$ (classe de la diagonale).

– Si $H^*(X) = \bigoplus_{i=0}^{2d} H^i(X)$ est une cohomologie de Weil, alors Π_i agit sur elle comme la projection de $H^i(X)$. (Dans cet article on prend $k = \mathbf{C}$ et $H^i(X)$ est la cohomologie de Betti.)

Soitent X et S des variétés projections sur $\mathbf{C}$, X lisse et $p : X \to S$ une application projective. Supposons la situation suivante :

$S^0 \subset S$ est un ensemble ouvert, régulier sur $\mathbf{C}$ tel que $\Sigma = S - S^0$, le complémentaire, soit un nombre fini de points et que p soit lisse sur S^0. Soient $X^0 = p^{-1}(S^0)$ et $p^0 : X^0 \to S^0$ la restriction de p. Le complémentaire $X - X^0$ est supposé être un diviseur avec croisements normaux. Il existe une résolution des singularités $p' : \widetilde{S} \to S$ (un isomorphisme sur S^0) tel que p se factorise par $\widetilde{S}$. Nous supposons que $\widetilde{S} - S^0$ soit un diviseur avec croisements normaux. Soit $d = \dim X - \dim S$.

THÉORÈME. – *Soit $p : X \to S$ vérifiant les hypothéses suivantes* :

(i) *Les composants irréducibles de $\widetilde{S} - S^0$ sont des variétés toriques projectives, lisses.*

(ii) *Les composants de $X - X^0$ sont des variétés toriques projectives, lisses.*

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(iii) *La variété* X^0/S^0 *a une décomposition de Chow–Künneth relative.*
(iv) $\widetilde{S}$ *a une décomposition de Chow–Künneth* (*c'est-à-dire qu'elle a des projecteurs de Chow–Künneth*).
(v) *Si* t *est un point de* S^0, *l'application naturelle*

$$\mathrm{CH}^r(X) \to H_B^{2r}\big(X_t(\mathbf{C}), \mathbf{Q}\big)^{\pi_1^{\mathrm{top}}(S^0,t)}, \quad (0 \leqslant r \leqslant d = \dim X - \dim S),$$

est surjective. (*La cible est la partie invariante sous l'action du groupe topologique fondemental de* S^0.)
(vi) *Pour* i *impair,* $H_B^i(X_t(\mathbf{C}), \mathbf{Q})^{\pi_1^{\mathrm{top}}(S^0,t)} = 0$.
(vii) *Si* $\mathcal{V}$ *est une représentation non constante, irréductible de* $\pi_1^{\mathrm{top}}(S^0, t)$ *et* $\mathcal{V}$ *le système local correspondant sur* S^0, *alors* $H^q(S^0, \mathcal{V}) = 0$ *si* $q \neq \dim S$.

Alors, sous ces hypothèses, X *a des projecteurs de Chow–Künneth.*

Let X be a smooth projective variety of dimension d over a field k. The Chow–Künneth projectors of X [13] are elements Π_i, $i = 0, \dots, 2d$, in the correspondence ring $\mathrm{CH}_d(X \times X)$, satisfying the following conditions:

– The Π_i are mutually orthogonal projectors and $\sum_i \Pi_i = \Delta_X$ (the class of the diagonal).
– If $H^*(X) = \bigoplus_{i=0}^{2d} H^i(X)$ is a Weil cohomology, then Π_i acts on it as the projection to $H^i(X)$. (In this paper we take $k = \mathbf{C}$ and $H^i(X)$ to be Betti cohomology.)

It is conjectured that Chow–Künneth projectors exist for any X. (The existence is known for curves, surfaces [12], and abelian varieties.) Note that the cohomology classes Π_i are the Künneth components of the diagonal class in $H^{2d}(X \times X)$. So the conjecture is stronger than a conjecture of Grothendieck – Conjecture (C) in his standard conjectures, [10,11] – that the Künneth components of the diagonal be the classes of algebraic cycles. Further, the projectors Π_i should satisfy additional properties [13].

For smooth projective maps between smooth quasi-projective varieties, one can analogously define the notion of relative Chow–Künneth projectors. They exist for abelian schemes [5].

One may further consider relative Chow–Künneth projectors for any (not necessarily smooth) projective map $p : X \to S$, with X smooth over $\mathbf{C}$, as in [4]. They are projectors in the ring $\mathrm{CH}_{\dim X}(X \times_S X)$ that give the decomposition of the derived image $\mathbf{R}p_*\mathbf{Q}_X$ into the sum of shifts of intersection complexes of local systems on strata of S.

Under certain, rather strong, assumptions on the fibers of p, we show such relative projectors exist, Theorem 1. With additional assumptions, it can be proven X has absolute Chow–Künneth projectors, Theorem 2. The assumptions for Theorem 2 are satisfied for Kuga–Shimura families over a Hilbert modular variety. We give only an outline of the proofs of Theorems 1 and 2; details will be published elsewhere.

We note the case of elliptic modular varieties had been studied in [8,9].

1. Relative Chow–Künneth projectors

Throughout this paper, for a variety X over $\mathbf{C}$, $\mathrm{CH}_k(X) := A_k(X) \otimes \mathbf{Q}$ is its rational Chow group, where $A_k(X)$ is as defined as in [6], and $H^i(X) := H_B^i(X(\mathbf{C}), \mathbf{Q})$ (Betti cohomology with rational coefficients). Also, we will consider only complexes of sheaves of $\mathbf{Q}$-vector spaces.

Let X and S be a quasi-projective varieties over $\mathbf{C}$ and assume X smooth. Let $p : X \to S$ be a projective surjective map. Assume

(i) there is a finite set of points $\Sigma \subset S$ such that p is smooth over $S - \Sigma$;
(ii) $Y = (p^{-1}\Sigma)_{\mathrm{red}}$ is a divisor with normal crossings;
(iii) all the irreducible components Y_j of Y are toric varieties;
(iv) there exist relative Chow–Künneth projectors for the map $p^0 := p|_{X^0} : X^0 \to S^0$ where $S^0 = S - \Sigma$ and $X^0 = p^{-1}(S^0)$. Namely there is an orthogonal set of projectors $(P^i)^0 \in \mathrm{CH}_{\dim X}(X^0 \times_{S^0} X^0)$ with $\sum (P^i)^0 = 1$, such that $(P^i)^0$ acts on $R^j p^0_* \mathbf{Q}_{X^0}$ as identity if $j = i$, and as zero if $j \neq i$.

Let $\widetilde{Y} \to Y$ be the normalization. The assumption (iii) implies $\mathrm{CH}_k(\widetilde{Y} \times_S \widetilde{Y}) \xrightarrow{\sim} H_{2k}(\widetilde{Y} \times_S \widetilde{Y})$. Let $q : \widetilde{Y} \to \Sigma$ and $\iota : \widetilde{Y} \to X$ be the induced maps. Let $i : \Sigma \to S$ be the closed immersion, and $j : S^0 \to S$ the open immersion:

$$\begin{tikzcd}
\widetilde{Y} \arrow[r, "\iota"] \arrow[d, "q"] & X \arrow[d, "p"] & X^0 \arrow[l] \arrow[d, "p^0"] \\
\Sigma \arrow[r, "i"] & S & S^0 \arrow[l, "j"']
\end{tikzcd}$$

The group $\mathrm{CH}_{\dim X}(X \times_S X)$ has a natural ring structure, and acts on the derived image $\mathbf{R}p_*\mathbf{Q}_X$. For a projector $\Pi \in \mathrm{CH}_{\dim X}(X \times_S X)$, the corresponding endomorphism $\Pi_* \in \mathrm{End}(\mathbf{R}p_*\mathbf{Q}_X)$ is a projector; we denote by $\Pi_*\mathbf{R}p_*\mathbf{Q}_X$ its image. For a local system $\mathcal{V}$ on S^0, $IC_S(\mathcal{V})$ denotes its intersection complex [7].

THEOREM 1. – *There exist local systems (indexed by a finite set of integers j) $\mathcal{V}_S^j$ on S^0, $\mathcal{V}_\Sigma^j$ on Σ, respectively, a finite set of mutually orthogonal projectors adding up to* 1 *in the ring* $\mathrm{CH}_{\dim X}(X \times_S X)$, $\{\Pi_S^j, \Pi_\Sigma^j\}$, *and isomorphisms* $(\Pi_S^j)_*\mathbf{R}p_*\mathbf{Q}_X \xrightarrow{\sim} IC_S(\mathcal{V}_S^j)[-j + \dim S]$ *and* $(\Pi_\Sigma^j)_*\mathbf{R}p_*\mathbf{Q}_X \xrightarrow{\sim} i_*\mathcal{V}_\Sigma^j[-j]$. *In other words, there exists an isomorphism* $\mathbf{R}p_*\mathbf{Q}_X \xrightarrow{\sim} \bigoplus IC_S(\mathcal{V}_S^j)[-j + \dim S] \oplus \bigoplus i_*\mathcal{V}_\Sigma^j[-j]$ *under which* $(\Pi_S^j)_*$ *and* $(\Pi_\Sigma^j)_*$ *are projections to the direct summands on the right-hand side.*

The projectors as stated are called *relative Chow–Künneth projectors*. Note the decomposition of $\mathbf{R}p_*\mathbf{Q}_X$ as above is known to exist in the derived category of $\mathbf{Q}$-sheaves $D^b(S)$ by [2,15].

2. Absolute Chow–Künneth projectors

Let X and S be projective varieties over $\mathbf{C}$, X smooth, and $p : X \to S$ a surjective projective map. Assume that we have the following situation:

$S^0 \subset S$ is an open set, smooth over $\mathbf{C}$, such that its complement $\Sigma = S - S^0$ is a finite set of points, and p is smooth over S^0. Let $X^0 = p^{-1}(S^0)$, and $p^0 : X^0 \to S^0$ be the restriction of p. The complement $X - X^0$ is assumed to be a divisor with normal crossings. There is a resolution of singularities $p' : \widetilde{S} \to S$ (an isomorphism over S^0) such that p factors through $\widetilde{S}$. We assume $\widetilde{S} - S^0$ is a divisor with normal crossings. Let $d = \dim X - \dim S$.

$$\begin{tikzcd}
X^0 \arrow[r] \arrow[d, "p^0"'] & X \arrow[d, "p"'] \arrow[dr] & \\
S^0 \arrow[r] & S & \widetilde{S} \arrow[l]
\end{tikzcd}$$

THEOREM 2. – *Let $p : X \to S$ satisfy the following assumptions.*

(i) *The irreducible components of $\widetilde{S} - S^0$ are smooth projective toric varieties.*
(ii) *The irreducible components of $X - X^0$ are smooth projective toric varieties.*
(iii) *The variety X^0/S^0 has a relative Chow–Künneth decomposition.*
(iv) *$\widetilde{S}$ has a Chow–Künneth decomposition (namely, it has Chow–Künneth projectors).*
(v) *If t is a point of S^0, the natural map* $\mathrm{CH}^r(X) \to H_B^{2r}(X_t(\mathbf{C}), \mathbf{Q})^{\pi_1^{\mathrm{top}}(S^0,t)}$ $(0 \leqslant r \leqslant d = \dim X - \dim S)$ *is surjective. (The target is the invariant part under the action of the topological fundamental group of S^0.)*
(vi) *For i odd,* $H_B^i(X_t(\mathbf{C}), \mathbf{Q})^{\pi_1^{\mathrm{top}}(S^0,t)} = 0$.
(vii) *If $\mathcal{V}$ is an irreducible, non-constant representation of $\pi_1^{\mathrm{top}}(S^0, t)$ and $\mathcal{V}$ the corresponding local system on S^0, then $H^q(S^0, \mathcal{V}) = 0$ if $q \neq \dim S$.*

Under these assumptions X has a Chow–Künneth decomposition.

3. Outline of proofs of Theorems 1 and 2

Let $CH\mathcal{M}(S)$ be the pseudo-abelian category of Chow motives over S. See [4] for details. If X is a smooth variety with a projective (not necessarily smooth) map $p: X \to S$, and $r \in \mathbf{Z}$, there is an associated object $h(X/S)(r)$ in $CH\mathcal{M}(S)$. If $q: Y \to S$ is another such, then $\mathrm{Hom}(h(X/S)(r), h(Y/S)(s)) = \mathrm{CH}_{\dim Y - s + r}(X \times_S Y)$. Given a map $f: X \to Y$ over S, one has morphisms $f^*: h(Y/S) \to h(X/S)$, and $f_*: h(X/S)(\dim X) \to h(Y/S)(\dim Y)$.

Under the assumption of Theorem 1, we have morphisms $h(\widetilde{Y}/S)(-1) \xrightarrow{\iota_*} h(X/S) \xrightarrow{\iota^*} h(\widetilde{Y}/S)$. We can show that there exists an element $\widetilde{\Pi}_\Sigma \in \mathrm{CH}_{\dim X}(\widetilde{Y} \times_S \widetilde{Y}) = \mathrm{Hom}(h(\widetilde{Y}/S), h(\widetilde{Y}/S)(-1))$ satisfying the following identities:

$$\iota^* \iota_* \widetilde{\Pi}_\Sigma \iota^* \iota_* = \iota^* \iota_*, \tag{1}$$

$$\widetilde{\Pi}_\Sigma \iota^* \iota_* \widetilde{\Pi}_\Sigma = \widetilde{\Pi}_\Sigma. \tag{2}$$

According to the decomposition $\mathrm{CH}_{\dim X}(\widetilde{Y} \times_S \widetilde{Y}) = \bigoplus_j \mathrm{Hom}(R^{j+2} q_* \mathbf{R}_{\widetilde{Y}}, R^j q_* \mathbf{R}_{\widetilde{Y}})$ we decompose $\widetilde{\Pi}_\Sigma = \bigoplus_j \widetilde{\Pi}_\Sigma^j$.

Let $\Pi_\Sigma^j := (\iota \times \iota)_* \widetilde{\Pi}_\Sigma^j = \iota_* \widetilde{\Pi}_\Sigma^j \iota^* \in \mathrm{CH}_{\dim X}(X \times_S X) = \mathrm{End}(h(X/S))$, and $\Pi_\Sigma = \iota_* \widetilde{\Pi}_\Sigma \iota^* = \sum \Pi_\Sigma^j$. The condition (2) implies that Π_Σ^j are mutually orthogonal projectors, one can prove, using (1), $\mathrm{Im}(\Pi_\Sigma^j)_* \mathbf{R} p_* \mathbf{Q}_X \xrightarrow{\sim} i_* \mathcal{V}_\Sigma^j[-j]$. Here the $\mathcal{V}_\Sigma^j$ are local systems on Σ such that $\mathbf{R} p_* \mathbf{Q}_X \xrightarrow{\sim} \bigoplus IC_S(\mathcal{V}_S^i)[-i + \dim S] \oplus \bigoplus i_* \mathcal{V}_\Sigma^i[-i]$ (the decomposition theorem, [2,15]).

Next let I be defined by the exact sequence $0 \to I \to \mathrm{CH}_{\dim X}(X \times_S X) \xrightarrow{\varphi} \mathrm{CH}_{\dim X}(X^0 \times_{S^0} X^0) \to 0$, where φ is the restriction. The group I is an ideal of $\mathrm{CH}_{\dim X}(X \times_S X)$. Let R be the two-sided orthogonal complement of Π_Σ, namely $R := (1 - \Pi_\Sigma) \circ \mathrm{CH}_{\dim X}(X \times_S X) \circ (1 - \Pi_\Sigma)$. It is a subring of $\mathrm{CH}_{\dim X}(X \times_S X)$ with identity $1 - \Pi_\Sigma$. One thus has an exact sequence $0 \to I \cap R \to R \xrightarrow{\varphi} \mathrm{CH}_{\dim X}(X^0 \times_{S^0} X^0) \to 0$. It can be shown, using (1) above, that the ideal I is nilpotent.

Take an orthogonal set of projectors $\Pi_{S^0}^j \in \mathrm{CH}_{\dim X}(X^0 \times_{S^0} X^0)$ adding up to 1 which gives rise to the decomposition on S^0: $\mathbf{R} p_* \mathbf{Q}_{X^0} = \bigoplus R^{j - \dim S} p_*^0 \mathbf{Q}_{X^0}[-j + \dim S]$. The $\Pi_{S^0}^j$ can be lifted to orthogonal projectors Π_S^j in R adding up to the identity $1 - \Pi_\Sigma$. The projectors thus obtained give rise to a decomposition as stated in Theorem 1.

For Theorem 2, we start with the relative Chow–Künneth projectors $\Pi_S^i(X/S)$ and $\Pi_\Sigma^i(X/S)$ in $\mathrm{CH}_{\dim X}(X \times_S X)$. Set $P^i = \Pi_S^{i + \dim S}(X/S)$, $P_\infty^i = \Pi_\Sigma^i(X/S)$.

For each r, using the assumption (v), we construct the projector P_{alg}^{2r}, which is a constituent of P^{2r} (namely $P^{2r} - P_{\mathrm{alg}}^{2r}$ is also a projector), so that its restriction $(P_{\mathrm{alg}}^{2r})^0 := (P_{\mathrm{alg}}^{2r})|S^0 \in \mathrm{CH}_{\dim X}(X^0 \times_{S^0} X^0)$ satisfies $(P_{\mathrm{alg}}^{2r})_*^0 R^{2r} p_*^0 \mathbf{Q}_{X^0} = (R^{2r} p_*^0 \mathbf{Q}_{X^0})^{\pi_1}$ the right-hand side being the constant local system of the π_1-invariant part. Let $P_{\mathrm{trans}}^{2r} = P^{2r} - P_{\mathrm{alg}}^{2r}$. The mutually orthogonal projectors adding to 1 in $\mathrm{CH}_{\dim X}(X \times_S X)$, P^{2r-1}, P_{alg}^{2r}, P_{trans}^{2r}, and P_∞^i remain so in $\mathrm{CH}_{\dim X}(X \times X)$ under the obvious ring homomorphism. We keep the same notation for their images.

We obtain the decomposition of the motive $M = h(X) \in CH\mathcal{M}(\mathbf{C})$ into the sum of the following four types of motives.

$$M^{2r-1} = \left(X, P^{2r-1}, 0\right) \quad (1 \leqslant r \leqslant d), \qquad M_{\mathrm{trans}}^{2r} = \left(X, P_{\mathrm{trans}}^{2r}, 0\right) \quad (0 \leqslant r \leqslant d),$$
$$M_{\mathrm{alg}}^{2r} = \left(X, P_{\mathrm{alg}}^{2r}, 0\right) \quad (0 \leqslant r \leqslant d), \quad \text{and} \quad M_\infty^i = \left(X, P_\infty^i, 0\right).$$

The first two and the last types have “pure” cohomology, i.e., cohomology in only one degree. To verify this we use the assumptions (vi) and (vii).

For the third type, we can show there is an isomorphism (for some m) $(X, P_{\mathrm{alg}}^{2r}, 0) \cong \bigoplus^m (\widetilde{S}, 1 - \Pi_\infty(\widetilde{S}/S), -r)$. Here $\Pi_\infty(\widetilde{S}/S)$ is the projector at infinity for the map $\widetilde{S} \to S$ (which also exists by applying Theorem 1 to it). By assumption (iv) we can split the motive on the right into motives with "pure" cohomology.

4. Applications: families over Hilbert modular varieties

Let F be totally real number field of degree d of degree $d = [F : \mathbf{Q}]$, let $\mathcal{O} = \mathcal{O}_F$ denote the ring of integers of F, and let $\Phi = (\phi^{(1)}, \ldots, \phi^{(d)}) : F \hookrightarrow \mathbf{R}^d$ be the embedding into $\mathbf{R}^d$ induced by the d embeddings of F into $\mathbf{R}$. Note that Φ induces an embedding $\mathrm{SL}_2(F) \hookrightarrow \mathrm{SL}_2(\mathbf{R})^d$, thus $\mathrm{SL}_2(F)$ acts on $\mathfrak{h}^d$, the product of d complex upper half-planes, by componentwise linear fractional transformation.

Let $\Gamma \subset \mathrm{SL}_2(F)$ be a neat congruence subgroup. It acts on $\mathfrak{h}^d$ freely and properly discontinuously on $\mathfrak{h}^d$. The quotient as a complex manifold $S^0 := \Gamma \backslash \mathfrak{h}^d$ has the structure of a smooth quasi-projective variety over $\mathbf{C}$. There is a projective variety S, the Satake–Baily–Borel compatification, obtained by adding a finite set of points (the cusps) to S^0; we refer to it as a Hilbert modular variety. As is well-known, there is a smooth compactification $\widetilde{S}$ of S^0 lying over S such that $\widetilde{S} - S^0$ is a divisor whose components are smooth projective toric varieties.

Let $L \subset F \times F$ be a projective $\mathcal{O}$-module of rank 2 such that, for all $g \in \Gamma$, $g \cdot L = L$. The semi-direct product $\Gamma \ltimes L$ acts on $\mathfrak{h}^d \times \mathbf{C}^d$ by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, {}^{\mathrm{t}}(\mu, \nu)\right)(\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \mu\tau + \nu}{c\tau + d}\right).$$

(Here the embedding Φ and the multi-indexing of everything are omitted from the notation.) The quotient $X^0 := (\Gamma \ltimes L) \backslash (\mathfrak{h}^d \times \mathbf{C}^d)$ is a quasi-projective variety that maps to S^0 by projection on the first factor; one thus has a family of d-dimensional abelian varieties with real multiplication by $\mathcal{O}$.

For each $m \geqslant 1$, let $X_m^0 := \overbrace{X^0 \times_{S^0} \times \cdots \times_{S^0} X^0}^{m\ \text{times}}$, the m-fold self-product of X^0 over S^0, and $p^0 = p_m^0 : X_m^0 \to S^0$ be the projection. By the method of toroidal compactifications, [1,14], we can take a smooth projective variety X_m containing X_m^0 as an open set, such that the complement is a divisor with components smooth toric varieties, and that p^0 extends to a map $X_m \to \widetilde{S}$. In other words, for $p : X_m \to S$, the assumptions (i), (ii) of Theorem 2 are satisfied.

Over S^0, p is an abelian scheme, so the assumption (iii) is also satisfied by [5]. The hypotheses (iv)–(vii) can be verified by the knowledge of cohomology of the Hilbert modular variety and the vanishing theorem (by Matsushima–Shimura) of cohomology of S^0 with coefficients in a local system.

Applying Theorem 2 one concludes the X_m has Chow–Künneth projectors. In particular, Conjecture (C) of Grothendieck holds for X_m.

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