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On a class of local systems associated to plane curves

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Abstract We study a class of local systems on the complement of a germ of irreducible plane curve. We exhibit local systems which by [8] give rise to regular holonomic $\mathcal{D}$-modules with characteristic variety the union of the zero section with the conormal of the curve. ***To cite this article: P.C. Silva, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 421–426.***

Sur une classe de systèmes locaux associés aux courbes planes

Résumé On étudie une classe de systèmes locaux sur le complément d'un germe de courbe irréductible plane. On présente des systèmes locaux qui par [8] correspondent à des $\mathcal{D}$-modules holonomes réguliers dont la variété caractéristique est l'union de la section nulle avec le conormal de la courbe. ***Pour citer cet article : P.C. Silva, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 421–426.***

Version française abrégée

Soit D un disque centré à l'origine de $\mathbb{C}$ de rayon suffisamment petit. Soit $(Y,0)$, $Y \subset X = D \times \mathbb{C}$, un germe de courbe irréductible plane, de cône tangent $y = 0$. Soient $a \in D \setminus \{0\}$ et $b \in \mathbb{C}$ tel que $|b| \gg 1$. On définit $L_a = \{a\} \times \mathbb{C}$. On définit $\Omega = L_a \cap Y$. Soit $(g(\omega))$, $\omega \in \Omega$, un système de générateurs de $\pi_1(L_a \setminus Y, (a,b))$ tels que chaque $g(\omega)$ est librement homotope à un petit lacet positif encerclant ω. Soient $\mathcal{L}$ un système local sur $X \setminus Y$, V sa fibre en (a,b) et $\rho : \pi_1(X \setminus Y, (a,b)) \to \mathbf{GL}(V)$ la représentation correspondante. On pose $V^g = \ker(\rho(g) - \mathbf{1}_V)$ pour tous g. D'après [8], on dit que $\mathcal{L}$ (ou ρ) est *hypergéométrique* (HG) si les conditions équivalentes (i), (ii) et (iii) sont vérifiées. La codimension de $V^{g(\omega)}$ dans V est appellée *la multiplicité de* $\mathcal{L}$ (*le long de* Y). Si $\operatorname{codim}_V V^{g(\omega)} = 1$, $\rho(g(\omega))$ est une pseudo-réflexion et le $\det(\rho(g(\omega)))$ est désigné *la valeur propre spéciale de* $\mathcal{L}$.

Supposons que $Y = \{y^n = x^m\}$, $m > n$. Soient $a_0, a_1 \in \pi_1(L_a \setminus \Omega, (a,b))$ convenables, tels que a_0 est librement homotope à un lacet positif autour de l'enveloppe convexe de Ω et a_1 est librement homotope à un petit lacet positif autour de $\omega \in \Omega$. Le groupe fondamental $\pi_1(X \setminus Y, (a,b))$ admet la présentation (1), où $\alpha, \beta \in \mathbb{Z}$ tels que $\alpha m = \beta n + 1$. Étant donnés $\lambda \in \mathbb{C}^*$ et E un ensemble de racines m-ièmes de l'unité, avec n éléments et produit 1, on définit $A, C \in \mathbf{GL}(n)$ par (2) et $\sum_{i=0}^n z_i \xi^{n-i}$ par $\lambda \prod_{\varepsilon \in E} (\xi - (-1)^{\beta(n+1)} \lambda^\beta \varepsilon^\beta)$.

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THÉORÈME 1. – *La correspondance que associe aux* λ *et* E *comme ci-dessus les représentations linéaires* $\rho_{\lambda,E}: G \to \mathbf{GL}(n)$, $\rho_{\lambda,E}(a_0) = -z_n(AC^{-1})^n$, $\rho_{\lambda,E}(a_1) = A$, *met en bijection les paires* (λ, E) *et les classes d'isomorphisme de représentations HG de multiplicité un.*

Supposons que Y a pour paires de Puiseux $(\tilde{m}_i, n_i)_{i=1,\dots,p}$, $\tilde{m}_1 > n_1$. Pour $k, l \in \mathbb{Z}$, $0 \leqslant l \leqslant k \leqslant p$, on définit $\nu_{l,k} = n_{l+1} \cdots n_k$. La courbe Y admet un développement de Puiseux $y = \sum_{j \geqslant \tilde{m}_1 \nu_{1,p}} a_j x^{j/\nu_{0,p}}$, par exemple (3). Soient Y_k, $k = 0, \dots, p$, les courbes définits par les troncations $y = \sum_{j=\tilde{m}_1\nu_{1,p}}^{\tilde{m}_k \nu_{k,p}} a_j x^{j/\nu_{0,p}}$. On définit $\tilde{m}_0 = 1$. On choisit ε suffisamment petit et on définit $Y_{k,\varepsilon}$, $k = 0, \dots, p$, par (4). L'inclusion $X \setminus Y_{k,\varepsilon} \subset X \setminus Y_k$ est une équivalence d'homotopie. De plus, $Y_{k,\varepsilon} \subset Y_{k-1,\varepsilon}$.

Soit $\omega_1 \in \Omega$ fixé. Pour $i = 0, \dots, p$, soit Ω_i l'ensemble des points de Ω qui appartient à la composante connexe de $L_a \cap Y_{i,\varepsilon}$ contenant ω_1. Soit $\prec$ une relation d'ordre total sur Ω telle que $\omega \prec \omega'$ si $\omega \in \Omega_i$ et $\omega' \in \Omega \setminus \Omega_i$. Soient $\omega_1 \prec \cdots \prec \omega_{\nu_{0,p}}$ les éléments de Ω. Soit $\mathrm{g}(\omega) \in \pi_1(L_a \setminus Y_{p,\varepsilon}, (a,b))$, $\omega \in \Omega$, representé par un lacet librement homotope, dans $L_a \setminus Y_{p,\varepsilon}$, à un petit lacet positif autour de ω. On définit $a_i = g(\omega_1) \cdots g(\omega_{\nu_{i,p}})$, $i = 0, \dots, p$. Supposons que a_i est representé par un lacet librement homotope, dans $L_a \setminus Y_{i,\varepsilon}$, à un lacet positif autour de Ω_i. Soit $b_{-1} = 1$. Pour $i = 1, \dots, p$, on définit $b_{i-1} = b_{i-2}^{n_i} a_{i-1}^{m_i}$ et $c_i = b_{i-2}^{\alpha_i} a_{i-1}^{\beta_i} a_i$, où $\alpha_i, \beta_i \in \mathbb{Z}$ vérifient $\alpha_i m_i = \beta_i n_i + 1$. D'après [10], p. 460, on peut supposer que $\pi_1(X \setminus Y_{k,\varepsilon}, (a,b))$ admet la présentation (5). De plus, le morphisme évident $G_{k-1} \to G_k$ est injectif.

On définit $a_{p+1} = 1$ et $\Omega_{p+1} = \emptyset$. Soit $\mathcal{L}$ un système local sur $X \setminus Y_{p,\varepsilon}$ de rang $\mu\nu_{0,p}$ et multiplicité μ et soit $\rho: G_p \to \mathrm{Aut}(V)$ la représentation correspondante, où $V = \mathcal{L}_{(a,b)}$. Supposons que $V^{a_0} = (0)$. Alors $\mathcal{L}$ est HG. De plus, on obtient des filtrations p-opposées sur V, $F^{\cdot}$ et $\overline{F}^{\cdot}$, telles que $F^i = V^{a_i^{-1}a_0}$ et $\overline{F}^{p+1-i} = V^{a_i}$ pour $i = 0, \dots, p+1$. En particulier ρ est déterminée par le triple $(\rho(a_0), F^{\cdot}, \overline{F}^{\cdot})$.

On s'intéresse au problème inverse. Soit μ un entier positif. Soit V un espace vectoriel complexe de dimension $\mu\nu_{0,p}$. On considère un triple $(A, F^{\cdot}, \overline{F}^{\cdot})$ tel que $A \in \mathrm{Aut}(V)$ et $F^{\cdot}$, $\overline{F}^{\cdot}$ sont des filtrations p-opposées sur V vérifiant $\dim F^i = \mu\nu_{i,p}$ pour $i = 0, \dots, p$ et $\dim F^{p+i} = 0$, $\dim F^{-i} = \mu\nu_{0,p}$ pour $i \geqslant 1$. On définit $A_i \in \mathrm{Aut}(V)$, $i = 0, \dots, p$, par $A_i|_{F^i} = A|_{F^i}$ et $A_i|_{\overline{F}^{p+1-i}} = \mathbf{1}_{\overline{F}^{p+1-i}}$. Pour $j \geqslant i$, $A_j(\overline{F}^{p+1-i}) \subset \overline{F}^{p+1-i}$. Soient ${}^iA_j \in \mathrm{Aut}(F^i)$ les morphismes induits. On définit $B_{-1} = \mathbf{1}_V$, $B_{i-1} = B_{i-2}^{n_i} A_{i-1}^{m_i}$ et $C_i = B_{i-2}^{\alpha_i} A_{i-1}^{\beta_i} A_i$, $i = 1, \dots, p$.

LEMME 2. – *Supposons que* ${}^iA_i - \mathbf{1}_{F^i}$ *est inversible pour tous* i. *Si les images de* $[B_{i-1}, A_i]$ *et* $B_{i-1}^{\alpha_i} - C_i^{n_i}$ *sont contenues dans* $\overline{F}^{p+2-i}$, $i = 1, \dots, p$, *il y a une représentation de multiplicité* μ, $\rho: G_p \to \mathrm{Aut}(V)$, *telle que* $\rho(a_i) = A_i$ *pour* $i = 0, \dots, p$.

Les représentations données par le Lemme 2 seront appelées *non résonantes* (NR). Par (ii), les correspondants systèmes locaux NR sur $X \setminus Y_{p,\varepsilon}$ sont HG.

Soit $\mathcal{F}^{\lambda}(Y_{0,\varepsilon})$ la classe des systèmes locaux NR sur $X \setminus Y_{0,\varepsilon}$ de rang un et monodromie $\lambda \neq 1$. Soient $\mathcal{F}^{\lambda}(Y_{k,\varepsilon})$, $k = 1, \dots, p$, les classes des systèmes locaux NR $\mathcal{L}$ sur $X \setminus Y_{k,\varepsilon}$ de multiplicité un et valeur propre spéciale $\lambda \neq 1$, tels que $\mathcal{L}|_{X \setminus Y_{k-1,\varepsilon}} = \bigoplus_{i=0}^{n_k-1} \mathcal{L}_i$, où $\mathcal{L}_i \in \mathcal{F}^{\lambda_i}(Y_{k-1,\varepsilon})$ pour quelque λ_i.

THÉORÈME 3. – *Étant donnés des systèmes locaux* $\mathcal{L}_i \in \mathcal{F}^{\lambda_i}(Y_{k-1,\varepsilon})$, $i = 0, \dots, n_k - 1$, $\bigoplus_i \mathcal{L}_i$ *admet une extension* $\mathcal{L} \in \mathcal{F}^{\lambda}(Y_{k,\varepsilon})$ *si et seulement si* $\lambda_i = (-1)^{n_k+1}\lambda\varepsilon_i$, *où* $\lambda \neq 1$ *et les* ε_i *constituent un ensemble de racines* $\tilde{m}_k$*-ièmes de l'unité avec* n_k *éléments et produit un.*

COROLLAIRE 4. – *Soit* X *un ouvert connexe et simplement connexe de* $\mathbb{C}^2$. *Soit* (Y, q), $q \in X$, *un germe de courbe irréductible plane. Alors il y a un germe de* $\mathcal{D}$*-module holonome régulier de variété caractéristique* $T_Y^*X \cup T_X^*X$.

Dans le cas où Y est un cusp on peut trouver dans [9], Théorème 7, la classification des $\mathcal{D}$-modules holonomes réguliers $\mathcal{M}$, de variété caractéristique $T_Y^*X \cup T_X^*X$ et multiplicité un le long de T_Y^*X, tels que $\mathrm{DR}(\mathcal{M})|_{X \setminus Y}$ soit un système local hypergéométrique.

Let D be a small disc centered at the origin. Let $(Y,0)$, $Y \subset X = D \times \mathbb{C}$, be a germ of irreducible plane curve with tangent cone $y = 0$. Take $a \in D \setminus \{0\}$ and $b \in \mathbb{C}$ s.t. $|b| \gg 1$. Set $L_a = \{a\} \times \mathbb{C}$. Set $\Omega = L_a \cap Y$. Let $(g(\omega))$, $\omega \in \Omega$, be a system of generators of $\pi_1(L_a \setminus Y, (a,b))$ s.t. each $g(\omega)$ is freely homotopic to a small positive loop around ω. Let $\mathcal{L}$ be a local system on $X \setminus Y$. Set $V = \mathcal{L}_{(a,b)}$. Let $\rho : \pi_1(X \setminus Y, (a,b)) \to \mathbf{GL}(V)$ be the monodromy of $\mathcal{L}$. Set $V^g = \ker(\rho(g) - \mathbf{1}_V)$ for all g. Following Neto, [8], we call $\mathcal{L}$ (or ρ) *hypergeometric* (HG) if the following equivalent conditions hold:

(i) There is a (unique) decomposition $V = \bigoplus_{\omega\in\Omega} U_\omega$ s.t. $V^{g(\omega)} = \bigoplus_{\sigma\neq\omega} U_\sigma$,

(ii) $\dim V = \sum_{\omega\in\Omega} \operatorname{codim}_V V^{g(\omega)}$ and $\bigcap_{\omega\in\Omega} V^{g(\omega)} = (0)$,

(iii) $H^*(L_a, j_*(\mathcal{L}|_{L_a\setminus Y})) = 0$, where $j : L_a \setminus Y \hookrightarrow L_a$ is the inclusion map.

These representations arise in several different situations (cf. [1,2,5]).

By (ii) we can replace the system $(g(\omega))$ by a system of generators of $\pi_1(X \setminus Y, (a,b))$, $(g'(\omega))$, $\omega \in \Omega$, s.t. $g'(\omega) = h_\omega^{-1} g(\omega) h_\omega$ for some $h_\omega \in \pi_1(X \setminus Y, (a,b))$. If $\operatorname{codim}_V V^{g(\omega)} = 1$, $\rho(g(\omega))$ is a pseudo-reflection and $\det(\rho(g(\omega)))$ is called the *special eigenvalue of* $\mathcal{L}$. If Y is smooth, $\Omega = \{\omega\}$ and the HG condition reduces to $V^{g(\omega)} = (0)$.

Let $i : X \setminus Y \hookrightarrow X$ be the inclusion map. Given a local system $\mathcal{L}$ on $X \setminus Y$ we call *multiplicity of* $\mathcal{L}$ (*along* Y) the integer $\operatorname{mult}(\mathcal{L}) := \operatorname{rank}(\mathcal{L}) - \dim(i_*\mathcal{L})_z$, where z is a regular point of Y. The local system $\mathcal{L}$ is HG iff $(i_*\mathcal{L})_0 = 0$ and $\operatorname{rank}(\mathcal{L}) = \operatorname{mult}_0(Y)\operatorname{mult}(\mathcal{L})$.

Assume that $Y = \{y^n = x^m\}$, $m > n$. Let $a_0, a_1 \in \pi_1(L_a \setminus Y, (a,b))$ s.t. a_0 is freely homotopic to a positive loop around the convex envelop of Ω and a_1 is freely homotopic to a small positive loop around some $\omega \in \Omega$. Take $\alpha, \beta \in \mathbb{Z}$ s.t. $m\alpha = n\beta + 1$. For a suitable choice of the a_i's, the local fundamental group of Y has the presentation (5) for $k = 1$, $m_1 = m$, $n_1 = n$, $\alpha_1 = \alpha$ and $\beta_1 = \beta$, i.e.,

$$G = \left\langle a_0, a_1 \mid \left[a_0^m, a_1\right] = 1,\ a_0^{m\alpha} = \left(a_0^\beta a_1\right)^n \right\rangle \tag{1}$$

(where the composition of paths $\gamma\delta$ denotes δ followed by γ). Given $\lambda \in \mathbb{C}^*$ and E a set of m-roots of the unity, with n elements and product 1, define $\sum_{i=0}^n z_i \xi^{n-i} = \lambda \prod_{\varepsilon\in E}(\xi - (-1)^{\beta(n+1)}\lambda^\beta\varepsilon^\beta)$ and set

$$A = \begin{pmatrix} z_0 & & & \\ z_1 & 1 & & \\ & & \ddots & \\ z_{n-1} & & & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ -z_n & & & \end{pmatrix}. \tag{2}$$

THEOREM 1. – *The correspondence that associates to* λ *and* E *as above the linear representation* $\rho_{\lambda,E} : G \to \mathbf{GL}(n)$, $\rho_{\lambda,E}(a_0) = -z_n(AC^{-1})^n$, $\rho_{\lambda,E}(a_1) = A$, *induces a bijection between the pairs* (λ, E) *and the isomorphism classes of HG representations of multiplicity one.*

Proof. – Set $b_0 = a_0^m$ and $c_1 = a_0^\beta a_1$. Since CA^{-1} is the companion matrix of $\prod_{\varepsilon\in E}(\xi - (-1)^{\beta(n+1)}\lambda^\beta\varepsilon^\beta)$, $\rho(a_0)$ is semi-simple with eigenvalues $(-1)^{n+1}\lambda\varepsilon$, $\varepsilon \in E$. In particular $\rho(b_0) = (-1)^{(n+1)m}\lambda^m$. Since $(AC^{-1})^{n\beta}A = (AC^{-1})^{m\alpha}C$, $\rho(c_1) = C$. Hence $\rho(b_0^\alpha) = \rho(c_1^n)$. Therefore $\rho_{\lambda,E}$ is a linear representation. For $i = 0,\dots,n-1$, set $g_i = c_1^{-i}a_1c_1^i$. Then $a_0 = (a_1c_1^{-1})^n c_1^n = \prod_{i=0}^{n-1} g_i$. Since $(m,n) = 1$, $\operatorname{codim} V^{a_1} = 1$ and $\bigcap_{i=0}^{n-1} V^{g_i} = (0)$. Hence $\rho_{\lambda,E}$ is HG of multiplicity one.

Assume that $\rho : G \to \mathbf{GL}(V)$ is HG of multiplicity one. By (i) there is a system of generators of V, u_i, $i \in \mathbb{Z}$, s.t. $u_{i+n} = u_i$, $(u_0,\dots,u_{n-1})$ is a basis of V and $V^{g_i} = \operatorname{span}\langle u_j : j \not\equiv i \pmod n\rangle$. Since $\rho(b_0)(V^{g_i}) \subset V^{g_i}$ for all i, $\rho(b_0)(\mathbb{C}u_i) \subset \mathbb{C}u_i$. Since $\rho(c_1)(V^{g_{i+1}}) \subset V^{g_i}$ for all i, we can assume that the matrices of $\rho(a_1)$ and $\rho(c_1)$ w.r.t. $(u_0,\dots,u_{n-1})$, equal the matrices A and C respectively, with $z_0z_n \neq 0$. Hence $\rho(b_0) = k$ and $z_n = -k^\alpha$, for some $k \in \mathbb{C}^*$. There are ε_i, $i = 0,\dots,n-1$, s.t. $\varepsilon_i^m = 1$ and $\rho(a_0)$ is semi-simple with eigenvalues $k^{1/m}\varepsilon_i$. We can assume that $\prod_{i=0}^{n-1}\varepsilon_i = 1$ which determines $k^{1/m}$. Since $\rho(a_0^\beta) = \rho(c_1a_1^{-1})$ is a diagonalizable companion matrix, the ε_i's are pairwise distinct. Set $\lambda = (-1)^{n+1}k^{1/m}$. Since $\rho(a_1) = \rho(a_0^{-\beta}c_1)$, $\sum_{i=0}^n z_i\xi^{n-i} = \lambda\prod_{i=0}^{n-1}(\xi - (-1)^{\beta(n+1)}\lambda^\beta\varepsilon_i^\beta)$. □

Assume that Y has Puiseux pairs $(\tilde m_i, n_i)_{i=1,\dots,p}$, $\tilde m_1 > n_1$. For $k, l \in \mathbb{Z}$, $0 \leqslant l \leqslant k \leqslant p$, set $\nu_{l,k} = n_{l+1}\cdots n_k$. Then Y has a Puiseux expansion of the form $y = \sum_{j \geqslant \tilde m_1 \nu_{1,p}} a_j x^{j/\nu_{0,p}}$. For instance

$$\begin{aligned} y &= x^{\tilde m_1/n_1} + x^{\tilde m_2/n_1 n_2} + \cdots + x^{\tilde m_p/n_1\cdots n_p} \\ &= x^{m_1/n_1} + x^{m_1/n_1 + m_2/n_1 n_2} + \cdots + x^{m_1/n_1 + m_2/n_1 n_2 + \cdots + m_p/n_1\cdots n_p}. \end{aligned} \tag{3}$$

Let Y_k, $k = 0, \dots, p$, be the plane curve defined by the truncation $y = \sum_{j=\tilde m_1 \nu_{1,p}}^{\tilde m_k \nu_{k,p}} a_j x^{j/\nu_{0,p}}$. Set $\tilde m_0 = 1$. Fix ε small enough and define $Y_{k,\varepsilon}$, $k = 0, \dots, p$, by

$$\Bigl| y - \sum\nolimits_{j=\tilde m_1 \nu_{1,p}}^{\tilde m_k \nu_{k,p}} a_j x^{j/\nu_{0,p}} \Bigr| \leqslant \varepsilon \bigl| x^{\tilde m_k/\nu_{0,k}} \bigr|. \tag{4}$$

The inclusion $X \setminus Y_{k,\varepsilon} \subset X \setminus Y_k$ is an homotopy equivalence. Moreover, $Y_{k,\varepsilon} \subset Y_{k-1,\varepsilon}$.

Fix some $\omega_1 \in \Omega$. For $i = 0, \dots, p$, let Ω_i be the set of points of Ω lying inside the connected component of $L_a \cap Y_{i,\varepsilon}$ that contains ω_1. Let $\prec$ be a total ordering on Ω s.t. $\omega \prec \omega'$ if $\omega \in \Omega_i$ and $\omega' \in \Omega \setminus \Omega_i$. Let $\omega_1 \prec \cdots \prec \omega_{\nu_{0,p}}$ be the elements of Ω. Let $g(\omega) \in \pi_1(L_a \setminus Y_{p,\varepsilon}, (a,b))$, $\omega \in \Omega$, be represented by a loop freely homotopic, in $L_a \setminus Y_{p,\varepsilon}$, to a small positive loop around ω. Set $a_i = g(\omega_1)\cdots g(\omega_{\nu_{i,p}})$ for $i = 0, \dots, p$. Assume that a_i is represented by a loop freely homotopic, in $L_a \setminus Y_{i,\varepsilon}$, to a positive loop around Ω_i. Set $b_{-1} = 1$. For $i = 1, \dots, p$, set $b_{i-1} = b_{i-2}^{n_i} a_{i-1}^{m_i}$, $c_i = b_{i-2}^{\alpha_i} a_{i-1}^{\beta_i} a_i$, where $\alpha_i, \beta_i \in \mathbb{Z}$ s.t. $\alpha_i m_i = \beta_i n_i + 1$. From [10], p. 460, we can assume that $\pi_1(X \setminus Y_{k,\varepsilon}, (a,b))$ has the presentation

$$G_k = \bigl\langle a_0, \dots, a_k \mid [b_{i-1}, a_i] = 1,\ c_i^{n_i} = b_{i-1}^{\alpha_i},\ i = 1, \dots, k \bigr\rangle, \quad k = 0, \dots, p. \tag{5}$$

Since $[b_{i-1}, c_i] = 1$ and $b_{i-2}^{\alpha_i + n_i} a_{i-1}^{\beta_i + m_i} a_i = b_{i-1} c_i$, G_k does not depend on the (α_i, β_i)'s. Moreover, the obvious morphism $G_{k-1} \to G_k$ is injective and induces an isomorphism between G_k and the free product with amalgamated subgroup of G_{k-1} by an infinite cyclic group.

Set $a_{p+1} = 1$ and set $\Omega_{p+1} = \emptyset$. Let $\mathcal{L}$ be a local system on $X \setminus Y_{p,\varepsilon}$ of rank $\mu\nu_{0,p}$ and multiplicity μ. Set $V = \mathcal{L}_{(a,b)}$. Let $\rho : G_p \to \mathrm{Aut}(V)$ be the monodromy representation of $\mathcal{L}$. Assume that $V^{a_0} = (0)$. By (ii) $\mathcal{L}$ is HG. Since $a_i = \prod_{\omega \in \Omega_i} g(\omega)$ and $a_i^{-1} a_0 = \prod_{\omega \in \Omega \setminus \Omega_i} g(\omega)$, $\bigcap_{\omega \in \Omega_i} V^{g(\omega)} \subset V^{a_i}$ and $\bigcap_{\omega \in \Omega \setminus \Omega_i} V^{g(\omega)} \subset V^{a_i^{-1} a_0}$. Since $\operatorname{codim} \cap_j V_j \leqslant \sum_j \operatorname{codim} V_j$ for any family of linear subspaces V_j of V, $\dim \bigcap_{\omega \in \Omega_i} V^{g(\omega)} \geqslant \mu |\Omega \setminus \Omega_i|$ and $\dim \bigcap_{\omega \in \Omega \setminus \Omega_i} V^{g(\omega)} \geqslant \mu |\Omega_i|$. Since $V^{a_i} \cap V^{a_i^{-1} a_0} = (0)$ for $i = 0, \dots, p+1$, we have p-opposed filtrations on V, $F^{\cdot}$ and $\overline{F}^{\cdot}$, s.t. $F^i = V^{a_i^{-1} a_0}$ and $\overline{F}^{p+1-i} = V^{a_i}$. The triple $(\rho(a_0), F^{\cdot}, \overline{F}^{\cdot})$ determines ρ. Moreover, the isomorphism class of $(F^{\cdot}, \overline{F}^{\cdot})$ only depends on the topology of Y and on the multiplicity of $\mathcal{L}$.

We shall consider now the inverse problem. Let μ be a positive integer. Let V be a complex vector space of dimension $\mu\nu_{0,p}$. Consider a triple $(A, F^{\cdot}, \overline{F}^{\cdot})$ s.t. $A \in \mathrm{Aut}(V)$ and $F^{\cdot}$, $\overline{F}^{\cdot}$ are p-opposed filtrations on V verifying $\dim F^i = \mu\nu_{i,p}$ for $i = 0, \dots, p$ and $\dim F^{p+i} = 0$, $\dim F^{-i} = \mu\nu_{0,p}$ for $i \geqslant 1$. Define $A_i \in \mathrm{Aut}(V)$, $i = 0, \dots, p$, by $A_i|_{F^i} = A|_{F^i}$ and $A_i|_{\overline{F}^{p+1-i}} = \mathbf{1}_{\overline{F}^{p+1-i}}$. For $j \geqslant i$, $A_j(\overline{F}^{p+1-i}) \subset \overline{F}^{p+1-i}$. Let ${}^i A_j \in \mathrm{Aut}(F^i)$, $j \geqslant i$, be the induced morphisms. Set $B_{-1} = \mathbf{1}_V$. For $i = 1, \dots, p$ set $B_{i-1} = B_{i-2}^{n_i} A_{i-1}^{m_i}$ and $C_i = B_{i-2}^{\alpha_i} A_{i-1}^{\beta_i} A_i$.

LEMMA 2. – *Assume that* ${}^i A_i - \mathbf{1}_{F^i}$ *is invertible for all* i. *If the images of the maps* $[B_{i-1}, A_i]$ *and* $B_{i-1}^{\alpha_i} - C_i^{n_i}$ *are contained in* $\overline{F}^{p+2-i}$, $i = 1, \dots, p$, *there is a representation of multiplicity* μ, $\rho : G_p \to \mathrm{Aut}(V)$, *s.t.* $\rho(a_i) = A_i$ *for all* i.

Proof. – We prove the result for $p = 2$. Let $p_1 : F^1 \oplus \overline{F}^2 \to F^1$ and $q_2 : \overline{F}^2 \to F^1 \oplus \overline{F}^2$ be, respectively, the canonical projection and the canonical injection. There are $\vartheta_i \in \mathrm{Hom}(F^1, \overline{F}^2)$, $i = 1, 2$, s.t. $A_i = {}^1 A_i \oplus \mathbf{1}_{\overline{F}^2} + q_2 \vartheta_i p_1$. Moreover, $\vartheta_i = \vartheta_1 ({}^1 A_1 - \mathbf{1}_{F^1})^{-1} ({}^1 A_i - \mathbf{1}_{F^1})$ for $i = 1, 2$. Since $A_0 - \mathbf{1}_{F^0}$ is invertible, $\overline{F}^2 = \ker(A_1 - \mathbf{1}_{F^0})$ and $F^1 = \ker(A_1^{-1} A_0 - \mathbf{1}_{F^0})$. By the hypothesis $[B_0, A_1] = 0$, $C_1^{n_1} = B_0^{\alpha_1}$ and the images of $[B_1, A_2]$, $B_1^{\alpha_2} - C_2^{n_2}$ are contained in $\overline{F}^2$. In particular, $B_0 = \omega \oplus \varpi$, where $\omega \in \mathrm{Aut}(F^1)$, $\varpi \in \mathrm{Aut}(\overline{F}^2)$.

Moreover, $[\omega, {}^1A_1] = 0$ and $\vartheta_1\omega = \varpi\vartheta_1$. Given $M \in \mathrm{Aut}(F^1)$, $N \in \mathrm{Aut}(\overline{F}^2)$ and $\varphi \in \mathrm{Hom}(F^1, \overline{F}^2)$ set $M \oplus_\varphi N = M \oplus N + q_2(\varphi M - N\varphi)p_1$. We have $B_0 = \omega \oplus_\varphi \varpi$ and $A_i = {}^1A_i \oplus_\varphi \mathbf{1}_{\overline{F}^2}$, $i = 1, 2$, where $\varphi = \vartheta_1({}^1A_1 - \mathbf{1}_{F^1})^{-1}$. Since $(M \oplus_\varphi N)(M' \oplus_\varphi N') = MM' \oplus_\varphi NN'$ for all $M, M' \in \mathrm{Aut}(F^1)$ and $N, N' \in \mathrm{Aut}(\overline{F}^2)$, the relations $[B_1, A_2] = 0$, $B_1^{\alpha_2} = C_2^{n_2}$ hold. □

The representations constructed in Lemma 2 shall be called *nonresonant* (NR). They are characterized by the fact that $\dim V = \mu\nu_{0,p}$ and $\mathrm{codim}\ker(\rho(a_i) - \mathbf{1}_V)^m = \mu\nu_{i,p}$ for $i = 0, \ldots, p$ and $m = 1, 2, \ldots$ By (ii) the corresponding NR local systems on $X \setminus Y_{p,\varepsilon}$ are HG. The restrictions of these NR local systems to $X \setminus Y_{k,\varepsilon}$, $k = 0, \ldots, p$, are NR of multiplicity $\mu\nu_{k,p}$.

Let $\mathcal{F}^\lambda(Y_{0,\varepsilon})$ be the class of (nonresonant) rank one local systems on $X \setminus Y_{0,\varepsilon}$ with monodromy $\lambda \neq 1$. Let $\mathcal{F}^\lambda(Y_{k,\varepsilon})$, $k = 1, \ldots, p$, be the class of (nonresonant) local systems $\mathcal{L}$ on $X \setminus Y_{k,\varepsilon}$ s.t. $\mathcal{L}$ has multiplicity one, $\mathcal{L}$ has special eigenvalue $\lambda \neq 1$ and $\mathcal{L}|_{X \setminus Y_{k-1,\varepsilon}} = \bigoplus_{i=0}^{n_k-1} \mathcal{L}_i$, where $\mathcal{L}_i \in \mathcal{F}^{\lambda_i}(Y_{k-1,\varepsilon})$ for some λ_i.

THEOREM 3. – *Given $\mathcal{L}_i \in \mathcal{F}^{\lambda_i}(Y_{k-1,\varepsilon})$, $i = 0, \ldots, n_k - 1$, $\bigoplus_i \mathcal{L}_i$ has an extension $\mathcal{L} \in \mathcal{F}^\lambda(Y_{k,\varepsilon})$ iff $\lambda_i = (-1)^{n_k+1}\lambda\varepsilon_i$, where $\lambda \neq 1$ and the ε_i's are pairwise distinct $\tilde{m}_k$-roots of the unity with product one.*

Proof. – For $k = 1$ the result follows from Theorem 1. Assume that $k \geqslant 2$.

Let $\rho_i : G_{k-1} \to \mathrm{Aut}(V_i)$, $i = 0, \ldots, n_k - 1$, be the monodromy of $\mathcal{L}_i$ where $V_i = \mathcal{L}_{i,(a,b)}$. Let $(\rho_i(a_0), F_i^{\cdot}, \overline{F}_i^{\cdot})$ be the corresponding triple defined by $F_i^j = V_i^{a_j^{-1}a_0}$ and $\overline{F}_i^{k-j} = V_i^{a_j}$, $j = 0, \ldots, k-1$. Given a $\mathbb{C}$-linear decomposition $\bigoplus_i F_i^{k-1} = F \oplus \overline{F}$ with $\dim F = 1$, we define k-opposed filtrations on $\bigoplus_i V_i$, $F^{\cdot}$, $\overline{F}^{\cdot}$ setting $F^j = \bigoplus_i F_i^j$ and $\overline{F}^{k-j+1} = \bigoplus_i \overline{F}_i^{k-j}$ if $j \neq k$, $F^k = F$ and $\overline{F}^1 = \overline{F} \oplus \bigoplus_i \overline{F}_i^1$. The representation $\bigoplus_i \rho_i$ extends to a NR representation $\rho : G_k \to \mathrm{Aut}(\bigoplus_i V_i)$ iff there is a decomposition $F^{k-1} = F \oplus \overline{F}$ as above s.t. $(\bigoplus_i \rho_i(a_0), F^{\cdot}, \overline{F}^{\cdot})$ determines a NR representation of G_k. Let $F^{k-1} = F \oplus \overline{F}$ be such decomposition. Define $A_j \in \mathrm{Aut}(\bigoplus_i V_i)$, $j = 0, \ldots, k$, by $A_j = \bigoplus_i \rho_i(a_j)$, $j = 0, \ldots, k-1$, $A_k|_{F^k} = A_{k-1}|_{F^k}$ and $A_k|_{\overline{F}^1} = \mathbf{1}_{\overline{F}^1}$. By Lemma 2 the triple $(A_0, F^{\cdot}, \overline{F}^{\cdot})$ determines a NR representation of G_k iff the images of the morphisms $[B_{k-1}, A_k]$ and $B_{k-1}^{\alpha_k} - C_k^{n_k}$ are contained in $\overline{F}^2$. The morphisms B_{k-2}, A_{k-1}, A_k leave invariant $\overline{F}^2$ inducing morphisms of F^{k-1}, $\widetilde{B}_{k-2}$, $\widetilde{A}_{k-1}$ and $\widetilde{A}_k$ respectively. Moreover, $\widetilde{A}_{k-1} = \bigoplus_{i=0}^{n_k-1} \lambda_i$. Hence the triple $(A_0, F^{\cdot}, \overline{F}^{\cdot})$ determines a NR representation of G_k iff

$$\big[\widetilde{B}_{k-2}^{n_k}\widetilde{A}_{k-1}^{m_k}, \widetilde{A}_k\big] = 0 \quad \text{and} \quad \big(\widetilde{B}_{k-2}^{n_k}\widetilde{A}_{k-1}^{m_k}\big)^{\alpha_k} = \big(\widetilde{B}_{k-2}^{\alpha_k}\widetilde{A}_{k-1}^{\beta_k}\widetilde{A}_k\big)^{n_k}. \tag{6}$$

Assume that $\widetilde{B}_{k-2} = \theta_{k-1}\widetilde{A}_{k-1}^{\tilde{m}_{k-1}}$ for some $\theta_{k-1} \in \mathbb{C}^*$. By the proof of Theorem 1 this condition holds for $k = 2$ with $\theta_1 = (-1)^{(n_1+1)\tilde{m}_1}$. Set $\widetilde{\beta}_k = \alpha_k\tilde{m}_{k-1} + \beta_k$. Since $\tilde{m}_k = n_k\tilde{m}_{k-1} + m_k$ and since $\alpha_k\tilde{m}_k = \widetilde{\beta}_k n_k + 1$, relations of (6) hold iff the relations $[\widetilde{A}_{k-1}^{\tilde{m}_k}, \widetilde{A}_k] = 0$ and $(\widetilde{A}_{k-1}^{\widetilde{\beta}_k}\widetilde{A}_k)^{n_k} = \widetilde{A}_{k-1}^{\alpha_k\tilde{m}_k}$ hold. These last relations are equivalent to say that $\widetilde{A}_{k-1}$ and $\widetilde{A}_k$ determine a NR representation of multiplicity one of the local fundamental group of a curve with one Puiseux pair $(\tilde{m}_k, n_k)$. The theorem now follows from Theorem 1 by induction procedure with $\theta_{k-1} = (-1)^{(n_{k-1}+1)\tilde{m}_{k-1}}\theta_{k-2}^{n_{k-1}}$, $k = 3, \ldots, p$. □

COROLLARY 4. – *Let X be a connected and simply connected open subset of $\mathbb{C}^2$. Let (Y, q) be a germ of an irreducible plane curve at some point $q \in X$. Then there is a germ of a regular holonomic $\mathcal{D}$-module with characteristic variety $T_Y^*X \cup T_X^*X$.*

Proof. – Let $j : X \setminus Y \hookrightarrow X \setminus \{q\}$ and $i : X \setminus \{q\} \hookrightarrow X$ be the inclusion morphisms. Let $\mathcal{L}$ be a local system on $X \setminus Y$. The constructible sheaf $i_*j_*\mathcal{L}$ verifies the co-support conditions relatively to the stratification $(X \setminus Y, Y \setminus \{q\}, \{q\})$ henceforth is perverse (see, for instance, [3], p. 7). Assume that $\mathcal{L}$ is hypergeometric. By (ii) $i_*j_*\mathcal{L} = i_!j_*\mathcal{L}$. Applying the arguments of [8], pp. 238 and 239, we conclude that $\mathrm{SS}(i_!j_*\mathcal{L}) \subset T_X^*X \cup T_Y^*X$. If $\mathcal{L} \neq 0$, the previous inclusion is an equality. By Theorem 3 and the Riemann–Hilbert correspondence (see, for instance, [4,6,7]) there is a regular holonomic $\mathcal{D}$-module $\mathcal{M}$ s.t. $\mathrm{DR}(\mathcal{M}) = i_!j_*\mathcal{L}$. Moreover, $\mathrm{Char}(\mathcal{M}) = T_X^*X \cup T_Y^*X$. □

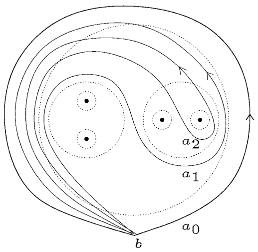

Figure 1. – Classes of loops for example 1.

Figure 1. – Classes des boueles.

In the case that Y is a cusp we can find in [9], Theorem 8, the classification of the regular holonomic $\mathcal{D}$-modules $\mathcal{M}$, with characteristic variety $T^*_Y X \cup T^*_X X$ and multiplicity one along $T^*_Y X$, s.t. $\mathrm{DR}(\mathcal{M})|_{X\setminus Y}$ is an hypergeometric local system.

Example 1. – Let Y_2 be the irreducible plane curve with Puiseux expansion $y = x^{5/2} + (1/2)x^{11/4}$. Assume that $a_i \in \pi_1((1\times\mathbb{C}) \setminus Y_{2,\varepsilon}, (1,b))$, $i = 0, 1, 2$, are the classes of the loops described in Fig. 1. Let $\mathcal{L} \in \mathcal{F}^{\lambda}(Y_{2,\varepsilon})$, $\mathcal{L}_i \in \mathcal{F}^{\lambda_i}(Y_{1,\varepsilon})$ and $\mathcal{L}_{i,j} \in \mathcal{F}^{\lambda_{i,j}}(Y_{0,\varepsilon})$, $i, j = 0, 1$, s.t. $\mathcal{L}|_{\mathbb{C}^2\setminus Y_{1,\varepsilon}} = \mathcal{L}_0 \oplus \mathcal{L}_1$ and $\mathcal{L}_i|_{\mathbb{C}^2\setminus Y_{0,\varepsilon}} = \mathcal{L}_{i,0} \oplus \mathcal{L}_{i,1}$. There are $\varepsilon_i, \varepsilon_{i,j}$, $i, j = 0, 1$, s.t. $\varepsilon_i^{11} = 1$, $\varepsilon_{i,j}^5 = 1$, $\varepsilon_0 = \varepsilon_1^{-1} \neq 1$, $\varepsilon_{i,0} = \varepsilon_{i,1}^{-1} \neq 1$, $\lambda_i = -\lambda\varepsilon_i$ and $\lambda_{i,j} = -\lambda_i\varepsilon_{i,j}$. Set $\zeta = \varepsilon_0$ and set $\vartheta_i = 1 + \varepsilon_{i,0} + \varepsilon_{i,1}$. Let $\rho : G_2 \to \mathrm{Aut}(V)$ be the monodromy of $\mathcal{L}$ and let $(\rho(a_0), F^{\cdot}, \overline{F}^{\cdot})$ be the corresponding triple. Let $f \in F^2 \setminus \{0\}$. Let $w \in F^1$ be the unique vector s.t. $w - \rho(c_2^{-1})(f) \in \overline{F}^2$. Then $(\rho(c_1^{-1})(f), \rho(c_1^{-1})(w))$ is a basis of $\overline{F}^2$ and the matrix of $\rho(a_0)$, w.r.t. the basis $(f, w, \rho(c_1^{-1})(f), \rho(c_1^{-1})(w))$, equals the matrix

$$\frac{1}{1-\zeta}\begin{pmatrix} (1-\zeta)\lambda & (\zeta^7-\zeta^5)\lambda^{-15} & (\vartheta_0\zeta^{10}-\vartheta_1\zeta^2)\lambda^{-1} & (\vartheta_1\zeta^7-\vartheta_0\zeta^5)\lambda^{-17} \\ (\zeta^5-\zeta^7)\lambda^{17} & (\zeta^2-\zeta^{10})\lambda & (\vartheta_0\zeta^5-\vartheta_1\zeta^7)\lambda^{15} & (\vartheta_1\zeta-\vartheta_0)\lambda^{-1} \\ (\vartheta_1\zeta^9-\vartheta_0\zeta^3)\lambda^3 & (\vartheta_0\zeta^9-\vartheta_1\zeta^3)\lambda^{-13} & (\vartheta_0\zeta-\vartheta_1)\lambda & (\vartheta_1\zeta^5-\vartheta_0\zeta^7)\lambda^{-15} \\ (\vartheta_1\zeta^3-\vartheta_0\zeta^9)\lambda^{19} & (\vartheta_0\zeta^4-\vartheta_1\zeta^8)\lambda^3 & (\vartheta_0\zeta^7-\vartheta_1\zeta^5)\lambda^{17} & (\vartheta_1\zeta^{10}-\vartheta_0\zeta^2)\lambda \end{pmatrix}.$$

The upper left 2×2-block of $\rho(a_0)$ determines the monodromy of a NR local system on the complement of an irreducible plane curve with one Puiseux pair $(11, 2)$ (see the proof of Theorem 3).

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References

[1] A. Dimca, A. Némethi, On the monodromy of complex polynomials, math.AG/9912072.
[2] E. Formanek, Braid group representations of low degree, Proc. London Math. Soc. 73 (3) (1996) 279–322.
[3] A. Galligo, M. Granger, Ph. Maisonobe, $\mathcal{D}$-Modules et faisceaux pervers dont le support singulier est un croisement normal, Ann. Inst. Fourier 35 (1985) 1–48.
[4] M. Kashiwara, The Riemann–Hilbert problem for holonomic systems, Publ. Res. Inst. Math. Sci. 20 (1984) 319–365.
[5] M. Kohno, Global Analysis in Linear Differential Equations, Kluwer Academic, 1999.
[6] Z. Mebkhout, Une équivalence de catégories, Comput. Math. 51 (1984) 51–62.
[7] Z. Mebkhout, Une autre équivalence de catégories, Comput. Math. 51 (1984) 63–88.
[8] O. Neto, A microlocal Riemann–Hilbert correspondence, Comput. Math. 127 (2001) 229–241.
[9] O. Neto, P.C. Silva, Holonomic Systems with solutions ramified along a cusp, C. R. Acad. Sci. Paris, Série I 335 (2002) 171–176.
[10] O. Zariski, On the topology of algebroid singularities, Amer. J. Math. 54 (1932) 453–465.