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Wavelet packets with uniform time-frequency localization

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Abstract We construct basic wavelet packets with uniformly bounded localization in both time and frequency. The corresponding orthonormal bases of wavelet packets are parametrized by dyadic segmentations obeying a local variation condition. ***To cite this article: L.F. Villemoes, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 793–796.***
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Paquets d'ondelettes avec localisation temps-fréquentielle uniforme

Résumé Nous construisons des paquets d'ondelettes de base uniformément bien localisés en temps et en fréquences. Les bases orthonormées correspondantes de paquets d'ondelettes sons parametrisées par des partitions dyadiques obeissants une condition de variation locale. ***Pour citer cet article : L.F. Villemoes, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 793–796.***
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1. Basic wavelet packets

Let g be a real valued continuous function with support included in $[-7\pi/6, 5\pi/6]$ and such that for $|\omega| \leqslant \pi/3$,

$$g\left(\tfrac{\pi}{2} - \omega\right)^2 + g\left(\tfrac{\pi}{2} + \omega\right)^2 = 1, \tag{1}$$

$$g\left(-2\omega - \tfrac{\pi}{2}\right) = g\left(\tfrac{\pi}{2} + \omega\right). \tag{2}$$

Define v as the inverse Fourier transform $v(t) = (2\pi)^{-1} \int g(\omega)\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}\omega$, and let

$$\psi_n(t) = 2\,\mathrm{Re}\left\{\exp\left(\mathrm{i}\pi\left(n + \tfrac{1}{2}\right)t\right) v\left((-1)^n\left(t - \tfrac{1}{2}\right)\right)\right\}. \tag{3}$$

THEOREM 1.1. – *The system $\psi_n(t-k)$, $k \in \mathbb{Z}$, $n = 0, 1, 2, \ldots$, forms an orthonormal basis for* $\mathrm{L}^2(\mathbb{R})$.

Proof. – The Fourier transform of ψ_n is given by

$$\widehat{\psi}_n(\omega) = \mathrm{e}^{-\mathrm{i}\omega/2}\left\{\mathrm{e}^{\mathrm{i}\alpha_n} g\left(\varepsilon_n\left(\omega - \pi\left(n + \tfrac{1}{2}\right)\right)\right) + \mathrm{e}^{-\mathrm{i}\alpha_n} g\left(\varepsilon_{n+1}\left(\omega + \pi\left(n + \tfrac{1}{2}\right)\right)\right)\right\}, \tag{4}$$

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where $\alpha_n = \frac{\pi}{2}(n+\frac{1}{2})$ and $\varepsilon_n = (-1)^n$. For $n \geqslant 1$ the hypotheses (1), (2) and disjoint support interiors give that $\|\psi_n\|^2 = (2\pi)^{-1}2\|g\|^2 = 1$. For $n=0$, observe that $|\widehat{\psi}_0(\omega)|^2 = |\mathrm{e}^{\mathrm{i}\alpha_0} g(\omega - \frac{\pi}{2}) + \mathrm{e}^{-\mathrm{i}\alpha_0} g(-\omega - \frac{\pi}{2})|^2 = g(\omega-\frac{\pi}{2})^2 + g(-\omega-\frac{\pi}{2})^2$, since $\mathrm{e}^{2\mathrm{i}\alpha_0}$ is purely imaginary, and $\|\psi_0\|^2 = 1$ follows as well.

Thus the system is normalized in L^2 and it is known from [4] that it suffices to show that for a.e. $\omega \in \mathbb{R}$,

$$\sum_{n=0}^{\infty} \widehat{\psi}_n(\omega + \pi l)\overline{\widehat{\psi}_n(\omega - \pi l)} = \delta_{0,l}, \quad l = 0, 1, 2, \ldots. \tag{5}$$

The case $l=0$ is easily verified by study of $|\widehat{\psi}_n(\omega)|^2$ as performed for the normalization. For $l \geqslant 1$, elementary computation and support considerations lead to

$$\widehat{\psi}_n(\omega+\pi l)\overline{\widehat{\psi}_n(\omega-\pi l)} = \mathrm{e}^{-\mathrm{i}\pi l}\,\mathrm{e}^{\mathrm{i}2\alpha_n} g\big(\varepsilon_n\big(\omega - \pi\big(n - l + \tfrac{1}{2}\big)\big)\big) g\big(\varepsilon_{n+1}\big(\omega + \pi\big(n - l + \tfrac{1}{2}\big)\big)\big),$$

which vanishes unless $n = l-1$ or $n = l$. Hence the sum (5) reduces to

$$\mathrm{e}^{-\mathrm{i}\pi/2} g\big(\varepsilon_{l-1}\big(\omega + \tfrac{\pi}{2}\big)\big) g\big(\varepsilon_l\big(\omega - \tfrac{\pi}{2}\big)\big) + \mathrm{e}^{\mathrm{i}\pi/2} g\big(\varepsilon_l\big(\omega - \tfrac{\pi}{2}\big)\big) g\big(\varepsilon_{l+1}\big(\omega + \tfrac{\pi}{2}\big)\big) = 0. \qquad \square$$

2. Wavelet packet bases

Let $\mathcal{D}$ be the collection of dyadic intervals $I_{j,n} = [2^j n, 2^j(n+1)[$, $j, n \in \mathbb{Z}$, and let $|I|$ denote the length of an interval I.

THEOREM 2.1. – *Assume $\mathcal{S} \subset \mathcal{D}$ is a collection of pairwise disjoint intervals such that $\bigcup_{\mathcal{S}} I = [0, +\infty[\setminus E$ where E is countable. Assume furthermore that for each $I = [a, b[\in \mathcal{S}$ there is an interval in $\mathcal{S}$ with left end point b and, if $a \neq 0$, also one with right end point a. Finally, suppose that for all adjacent pairs $I, J \in \mathcal{S}$ with $|I| > |J|$, it holds that $|I| = 2|J|$ and both intervals are contained in a dyadic interval $I_0 \in \mathcal{D}$ with $|I_0| = 2|I|$.*

Define $\psi_I(t) = 2^{j/2}\psi_n(2^j t)$ for $I = I_{n,j} \in \mathcal{D}$. Then $\psi_I(t - k|I|^{-1})$, $I \in \mathcal{S}$, $k \in \mathbb{Z}$, forms an orthonormal basis for $\mathrm{L}^2(\mathbb{R})$.

Proof. – For each $I = I_{n,j} \in \mathcal{S}$, it is clear by rescaling and Theorem 1.1 that $\psi_I(t - k|I|^{-1})$, $k \in \mathbb{Z}$, forms an orthonormal sequence in L^2, which spans a closed subspace V_I. It also follows that $V_I \perp V_J$ if $I \neq J$ for any $I, J \in \mathcal{S}$ with equal lengths. The main difficulty is to show that $V_I \perp V_J$ for all adjacent pairs $I, J \in \mathcal{S}$ with different lengths. Without loss of generality, let $I = [m, m+1[$ and $J = \frac{1}{2}[n, n+1[$ where $n = 2m+2$ if m is even and $n+1 = 2m$ if m is odd. In both cases, straightforward computations lead to

$$p(\omega) = \widehat{\psi}_n(2\omega)\overline{\widehat{\psi}_m(\omega)} = \mathrm{e}^{-\mathrm{i}\omega/2}\big\{\mathrm{e}^{\mathrm{i}(\alpha_n - \alpha_m)} b(\omega - \pi\kappa_m) + \mathrm{e}^{-\mathrm{i}(\alpha_n - \alpha_m)} b(\omega + \pi\kappa_m)\big\},$$

where $\alpha_n = \frac{\pi}{2}(n+\frac{1}{2})$, $b(x) = g(\frac{\pi}{2}+x)g(\frac{\pi}{2}-x)$ is a symmetric bump function and $\kappa_m = m + \frac{1}{2}(1+(-1)^m)$ is the meeting point of the intervals. Observe that κ_m is an odd integer and 2π-periodization of $p(\omega)$ yields

$$\sum_{l\in\mathbb{Z}} p(\omega + 2\pi l) = \mathrm{e}^{-\mathrm{i}\omega/2} \sum_{l\in\mathbb{Z}} \big((-1)^l\,\mathrm{e}^{\mathrm{i}\frac{\pi}{2}(n-m)} + (-1)^{l-\kappa_m} e^{-\mathrm{i}\frac{\pi}{2}(n-m)}\big) b(\omega + 2\pi l - \pi\kappa_m).$$

This sum vanishes since κ_m is odd and $n-m$ is even. Hence $\psi_n(t/2)$ is orthogonal to all integer translates of ψ_m, so $V_I \perp V_J$.

The orthogonality of V_I and V_J for nonadjacent $I, J \in \mathcal{S}$ is easily seen from support considerations of $\widehat{\psi}_I$ by using the assumption that all intervals in $\mathcal{S}$ have right neighbors. Let V be the closure of the orthogonal sum of all V_I, $I \in \mathcal{S}$. It remains to prove that $\mathrm{L}^2 \subset V$.

Given a positive odd integer κ, consider the orthogonal sum $H_- \oplus H_+$ corresponding to the segmentation $\frac{1}{2}[n, n+1[$, $n+1 \leqslant 2\kappa$, $[m, m+1]$, $m \geqslant \kappa$. Then, by Theorem 1.1, the orthogonal complements $H_+^\perp$ and $H_-^\perp$ have bases determined by the segmentations $[m, m+1]$, $m+1 \leqslant \kappa$, and $\frac{1}{2}[n, n+1[$, $n \geqslant 2\kappa$, respectively. Since these complements are orthogonal, it follows that $H_- \oplus H_+ = H_+^\perp \oplus H_-^\perp = L^2$. By repeating this argument, bases can be contructed corresponding to constant interval size extensions of all adjacent triples I^-, I, I^+ in $\mathcal{S}$. Due to the support properties of $\widehat{\psi}_I$ we infer that any $f \in \mathrm{L}^2$ with $\operatorname{supp}(\widehat{f}) \subset \pi(\overline{I} \cup (-\overline{I}))$ is contained in V. However, since $\bigcup_{\mathcal{S}} I = [0, +\infty[\setminus E$, it follows that $\mathrm{L}^2 \subset V$. □

3. Recursion

There exist continuous 2π-periodic symbols $m_0(\omega)$, $m_1(\omega)$ and $m_{r,\nu}(\omega)$, $r \in \{-2, -1, 0, 1\}$, $\nu \in \{-1, 0\}$, with $m_{-1-r,-1-\nu} = m_{r,\nu}$, such that

$$\widehat{\psi}_0(2\omega) = m_0(\omega)\widehat{\psi}_0(\omega), \qquad \widehat{\psi}_1(2\omega) = m_1(\omega)\widehat{\psi}_0(\omega), \tag{6}$$

$$\widehat{\psi}_{4q+r}(2\omega) = \sum_{\nu=-1}^{0} m_{r,\nu}(\omega)\widehat{\psi}_{2q+\nu}(\omega), \quad r \in \{-2, -1, 0, 1\},\ q = 1, 2, \ldots. \tag{7}$$

These facts follow easily from Theorem 2.1 and 2π-periodization of $\widehat{\psi}_{4q+r}(2\omega)\overline{\widehat{\psi}_{2q+\nu}(\omega)}$. All the symbols are explicit in terms of g, and (m_0, m_1) is a conjugate quadrature filter pair. For the symbols $m_{r,\nu}$, we have the matrix condition

$$\sum_{\varepsilon=0}^{1} \sum_{\nu=-1}^{0} m_{r,\nu}(\omega + \varepsilon\pi)\overline{m_{s,\nu}(\omega + \varepsilon\pi)} = \delta_{r,s}, \quad r, s \in \{-2, -1, 0, 1\}, \tag{8}$$

which is well known from the theory of refinable vectors, [5]. If g vanishes outside $[-9\pi/10, 7\pi/10]$ then $m_{1,-1} = 0$, but it is important that $m_{0,-1}$ is nontrivial. The split and merge property corresponding to (7) is that any pair of adjacent equal lengths dyadic intervals such that their union is not dyadic can be split in four half size intervals. This rule and its inverse both preserve the class of partitions of Theorem 2.1 and adaptive best basis algorithms can be developed easily for this structure following the ideas of [12].

4. Relation to standard wavelet packets

The standard basic wavelet packet construction in sequency order corresponds to the case where $m_{-1,-1} = m_{0,0} = m_0$ and $m_{-2,-1} = m_{1,0} = m_1$ and all other $m_{r,\nu} = 0$ in (7). In this case, the split and merge property is much more flexible, since any dyadic interval can be split in two. Unfortunately this freedom destroys the hope for uniform time-frequency localization. This has been very well studied in [3, 6,11,2,10]. Let us here furnish a proof in a simple formulation based on the identity

$$\left|\widehat{\psi}_m\left(2^{-j}\omega\right)\right|^2 = \sum_{n=2^j m}^{2^j(m+1)-1} \left|\widehat{\psi}_n(\omega)\right|^2 \tag{9}$$

which holds for all $j, m = 0, 1, 2, \ldots$ and follows directly from iterated merging. It holds true also for the nonstationary basic wavelet packets defined in [6,2], but *not* for the system (3). Assume all ψ_n are real valued and that there is a finite constant C such that

$$\int_0^\infty (\omega - n\pi)^2 \left|\widehat{\psi}_n(\omega)\right|^2 \mathrm{d}\omega \leqslant C^2, \quad n = 0, 1, 2, \ldots. \tag{10}$$

Then it is easy to show using Tchebycheff's inequality that the right-hand side $s(\omega)$ of (9) satisfies

$$\int_{N\pi-C}^{N\pi+C} s(\omega)\,\mathrm{d}\omega \geqslant \pi - 1 \quad \text{and} \quad \int_{(N+1)\pi+C^2}^{+\infty} s(\omega)\,\mathrm{d}\omega \leqslant \frac{1}{\pi}, \tag{11}$$

where $N = 2^j(m+1) - 1$ and we assume $2^j \geqslant C$. It follows that $\int_I s(\omega) - s(\omega + d)\,\mathrm{d}\omega \geqslant \pi - 1 - \frac{1}{\pi} > 0$, where $I = [N\pi - C, N\pi + C]$ and $d = \pi + C^2 + C$. Now, since (9) implies $s(\omega) = |\widehat{\psi}_m(2^{-j}\omega)|^2$ we find by letting $j \to \infty$ that $\widehat{\psi}_m$ cannot be uniformly continuous, so $\psi_m \notin \mathrm{L}^1$. In other words, (9) and (10) ruin the chances for a good time localization.

On the other hand it is clear from (4) that the system (3) satisfies (10) and if $g(\omega)$ is smooth then the window function $v(t)$ has rapid decay at infinity, so the time localization is trivial. What is left open to explore is whether there exist compactly supported functions ψ_n solving (6) and (7) and inheriting a good time-frequency localization.

Let us stress that constructions very similar to the bases of Theorem 2.1 are well known for the case of wavelet bases from [8], and in the general case from [7,9], and [1]. However, none of these bases possess the wavelet packet type of structure described here.

Finally, Theorem 2.1 still holds true if one takes the imaginary part in (3) instead of the real part.

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