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Probabilités/*Probability Theory*

# Sharp bounds for transition probability densities of a class of diffusions

**Zhongmin Qian** [a], **Weian Zheng** [b,c]

[a] Laboratoire de statistique et probabilités CNRS, Université Paul-Sabatier, 118, route de Narbonne, 31062 Toulouse cedex 4, France

[b] Department of Statistics, East China Normal University, Shanghai, China

[c] Department of Mathematics, University of California, Irvine, CA 92697, USA

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**Abstract** Let $p_b(x,t,y)$ be the transition probability density of the one dimensional diffusion process $\mathrm{d}X_t = \mathrm{d}W_t + b(X_t)\,\mathrm{d}t$, where $|b(\cdot)|_\infty \leqslant 1$. We show that the upper and lower bounds of $p_b(x,t,y)$ are achieved for fixed $(x,t,y)$ when $b(z) = \mathrm{sgn}(y-z)$ and $b(z) = \mathrm{sgn}(z-y)$ respectively. Moreover, the precise bounds are given. ***To cite this article: Z. Qian, W. Zheng, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 953–957.***


**Bornes explicites pour la densité de la probabilité de transition d'un processus de diffusion**

**Résumé** Soit $p_b(x,t,y)$ la densité de la probabilité de transition du processus de diffusion $\mathrm{d}X_t = \mathrm{d}W_t + b(X_t)\,\mathrm{d}t$, où $|b(\cdot)|_\infty \leqslant 1$. Nous montrons que la borne supérieure (resp. la borne inférieure) de $p_b(x,t,y)$ est atteinte pour $(x,t,y)$ fixés quand $b(z) = \mathrm{sgn}(y-z)$ (resp. quand $b(z) = \mathrm{sgn}(z-y)$). Les bornes explicites sont présentées. ***Pour citer cet article : Z. Qian, W. Zheng, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 953–957.***


## *Version française abrégée*

Considérons l'équation différentielle stochastique

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \mathrm{d}W_t, \tag{1}$$

où $b$ est une fonction mesurable bornée sur $\mathbb{R}$, et $(W_t)_{t\geqslant 0}$ est le mouvement brownien standard. Dans cette Note, nous prouvons que la densité de la probabilité de transition $p_b(x,t,y)$ de $(X_t)$ vérifie les bornes suivantes.

*E-mail addresses:* qian@math.ups-tlse.fr (Z. Qian); wzheng@uci.edu (W. Zheng).


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THÉORÈME 1. – *Soit $\beta = |b|_\infty < \infty$. Alors pour tous $x$, $y$ et $t > 0$*

$$\frac{1}{\sqrt{2\pi t}} \int_{|x-y|/\sqrt{t}}^{\infty} z\,\mathrm{e}^{-(z+\beta\sqrt{t})^2/2}\,\mathrm{d}z \leqslant p_b(x,t,y) \leqslant \frac{1}{\sqrt{2\pi t}} \int_{|x-y|/\sqrt{t}}^{\infty} z\,\mathrm{e}^{-(z-\beta\sqrt{t})^2/2}\,\mathrm{d}z. \tag{2}$$

De plus, nous avons le

THÉORÈME 2. – *Soit $y \in \mathbb{R}$ et $\beta$ une constante. Notons par $p_y^\beta(x,t,z)$ la densité de la probabilité* (*par rapport à la mesure de Lebesgue*) *associée au processus de diffusion*

$$\mathrm{d}X_t = \beta\,\mathrm{sgn}(y - X_t)\,\mathrm{d}t + \mathrm{d}W_t. \tag{3}$$

*Alors*

$$p_y^\beta(x,t,y) = \frac{1}{\sqrt{2\pi t}} \int_{|x-y|/\sqrt{t}}^{\infty} z\,\mathrm{e}^{-(z-\beta\sqrt{t})^2/2}\,\mathrm{d}z. \tag{4}$$

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Consider the following stochastic differential equation on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$:

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \mathrm{d}W_t, \tag{5}$$

where $b$ is a bounded measurable function on $\mathbb{R}$, and $(W_t)_{t\geqslant 0}$ is the standard Brownian motion. It is well known that there is a unique weak solution to (5) and its transition probability density (with respect to the Lebesgue measure) $p_b(x,t,y)$ is Hölder continuous in all variables (see, e.g., [1,6]). In this Note we provide the optimal upper and lower bounds for the transition probability density $p_b(x,t,y)$ in terms of the bound $|b|_\infty$. Let us state our main result as the following:

THEOREM 1. – *Assume $\beta = |b|_\infty < \infty$. Then for all $x$, $y$ and $t > 0$*

$$\frac{1}{\sqrt{2\pi t}} \int_{|x-y|/\sqrt{t}}^{\infty} z\,\mathrm{e}^{-(z+\beta\sqrt{t})^2/2}\,\mathrm{d}z \leqslant p_b(x,t,y) \leqslant \frac{1}{\sqrt{2\pi t}} \int_{|x-y|/\sqrt{t}}^{\infty} z\,\mathrm{e}^{-(z-\beta\sqrt{t})^2/2}\,\mathrm{d}z. \tag{6}$$

The bounds in (6) are optimal under the norm $|b|_\infty$, see Theorem 2. Our proof depends on the precise information on the transition probability densities for the extremal vector fields $b$ which we state as the following theorem, which has independent interest on its own.

THEOREM 2. – *Let $y \in \mathbb{R}$ and let $\beta$ be a constant. Denote by $p_y^\beta(x,t,z)$ the transition probability density* (*with respect to the Lebesgue measure*) *associated to the diffusion process*

$$\mathrm{d}X_t = \beta\,\mathrm{sgn}(y - X_t)\,\mathrm{d}t + \mathrm{d}W_t. \tag{7}$$

*Then*

$$p_y^\beta(x,t,y) = \frac{1}{\sqrt{2\pi t}} \int_{|x-y|/\sqrt{t}}^{\infty} z\,\mathrm{e}^{-(z-\beta\sqrt{t})^2/2}\,\mathrm{d}z. \tag{8}$$

*Proof.* – It is sufficient to consider the case where $|x-y| > 0$. The case where $|x-y| = 0$ follows the continuity of both sides of (8). Note that $p_y^\beta(x,t,z)$ is continuous in $(x,t,z)$. Therefore

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathbb{P}\big(|X_t - y| \leqslant \lambda\big) = p_y^\beta(x,t,y+\lambda) + p_y^\beta(x,t,y-\lambda)$$

for all $\lambda \geqslant 0$. In particular

$$p_y^\beta(x,t,y) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg|_{\lambda=0+}\mathbb{P}\big(|X_t - y| \leqslant \lambda\big).$$

Since

$$\mathrm{d}X_t = \beta\,\mathrm{sgn}(y - X_t)\,\mathrm{d}t + \mathrm{d}W_t; \quad X_0 = x,$$

by Tanaka's formula ([3], Proposition 3.6.8 or [5], Theorem VI.1.2)

$$\begin{aligned}|X_t - y| &= |x-y| + \int_0^t \mathrm{sgn}(X_s - y)\,\mathrm{d}X_s + L_t \\ &= |x-y| - \beta t + \int_0^t \mathrm{sgn}(X_s - y)\,\mathrm{d}W_s + L_t,\end{aligned}$$

where $(L_t)_{t\geqslant 0}$ is the local time of the process $(X_t)_{t\geqslant 0}$ at $y$. The local time $L_t$ is given by the Skorohod equation ([3], Lemma 3.6.14)

$$L_t = \max\left\{0, \sup_{s\leqslant t}\left(-|x-y| + \beta s - \int_0^s \mathrm{sgn}(X_u - y)\,\mathrm{d}W_u\right)\right\}.$$

Define $B_t = \beta t - \int_0^t \mathrm{sgn}(X_s - y)\,\mathrm{d}W_s$, and define the probability measure $\mathbb{Q}$ by

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{F}_t} = \exp\left(\int_0^t \beta\,\mathrm{sgn}(X_s - y)\,\mathrm{d}W_s - \frac{\beta^2 t}{2}\right).$$

Then by the Cameron–Martin formula, under $\mathbb{Q}$, $(B_t)_{t\geqslant 0}$ is a standard Brownian motion with initial value zero. Let $M_t = \sup_{s\leqslant t} B_s$ be the maximum process of $B_t$. Then we have

$$|X_t - y| = |x-y| - B_t + L_t$$

and

$$L_t = \max\big\{0, -|x-y| + M_t\big\},$$

so that

$$|X_t - y| = -B_t + \max\big\{|x-y|, M_t\big\}.$$

Since $(B_t, M_t)$ possesses the joint distribution

$$\mathbb{Q}(B_t \in \mathrm{d}a, M_t \in \mathrm{d}b) = \frac{2(2b-a)}{\sqrt{2\pi t^3}}\exp\left\{-\frac{(2b-a)^2}{(2t)}\right\}\mathrm{d}a\,\mathrm{d}b$$

on $a \leqslant b$ and $b \geqslant 0$ (see [3], formula (2.8.2)), therefore when $0 < \lambda < |x-y|$

$$\begin{aligned}\mathbb{P}\big(|X_t - y| \leqslant \lambda\big) &= \mathbb{Q}\left(\exp\left(\beta B_t - \frac{\beta^2 t}{2}\right) : |X_t - y| \leqslant \lambda\right) \\ &= \mathbb{Q}\left(\exp\left(\beta B_t - \frac{\beta^2 t}{2}\right) : -B_t + \max\big\{|x-y|, M_t\big\} \leqslant \lambda\right) \\ &= \iint_{\substack{a\leqslant b;\ b\geqslant 0 \\ -a+\max\{|x-y|,b\}\leqslant\lambda}} \mathrm{e}^{\beta a - \beta^2 t/2}\frac{2(2b-a)}{\sqrt{2\pi t^3}}\,\mathrm{e}^{-(2b-a)^2/(2t)}\,\mathrm{d}a\,\mathrm{d}b\end{aligned}$$

$$
\begin{aligned}
&= \int_{|x-y|-\lambda}^{+\infty} \int_{a}^{a+\lambda} \mathrm{e}^{\beta a-\beta^2 t/2} \frac{2(2b-a)}{\sqrt{2\pi t^3}} \, \mathrm{e}^{-(2b-a)^2/(2t)} \, \mathrm{d}b \, \mathrm{d}a \\
&= \frac{1}{\sqrt{2\pi t}} \int_{|x-y|-\lambda}^{+\infty} \mathrm{e}^{\beta a-\beta^2 t/2} \big(\mathrm{e}^{-a^2/(2t)} - \mathrm{e}^{-(a+2\lambda)^2/(2t)}\big) \, \mathrm{d}a.
\end{aligned}
$$

Taking derivative in $\lambda$ we thus obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\lambda} \mathbb{P}\big(|X_t - y| \leqslant \lambda\big) &= \frac{1}{\sqrt{2\pi t}} \, \mathrm{e}^{\beta(|x-y|-\lambda)-\beta^2 t/2} \big(\mathrm{e}^{-(|x-y|-\lambda)^2/(2t)} - \mathrm{e}^{-(|x-y|+\lambda)^2/(2t)}\big) \\
&\quad + \frac{\mathrm{e}^{-\beta^2 t/2}}{\sqrt{2\pi t}} \int_{|x-y|-\lambda}^{+\infty} \mathrm{e}^{-(a+2\lambda)^2/(2t)+\beta a} \frac{2(a+2\lambda)}{t} \, \mathrm{d}a.
\end{aligned}
$$

Letting $\lambda \downarrow 0$ we hence establish

$$
p_y^{\beta}(x,t,y) = \frac{\mathrm{e}^{-\beta^2 t/2}}{\sqrt{2\pi t}} \int_{|x-y|/\sqrt{t}}^{+\infty} z \, \mathrm{e}^{-z^2/2+\sqrt{t}\beta z} \, \mathrm{d}z. \qquad \square
$$

Now we are in a position to prove Theorem 1. Without losing generality we may assume that $|b|_\infty = 1$, and use $p_y^{+}(x,t,z)$ (resp. $p_y^{-}(x,t,z)$) to denote $p_y^{\beta}(x,t,z)$ when $\beta = 1$ (resp. when $\beta = -1$). Since

$$
\frac{\partial}{\partial x} p_y^{\beta}(x,t,y) = -\frac{x-y}{t\sqrt{2\pi t}} \, \mathrm{e}^{-(x-y)^2/(2t)-\beta^2 t/2+\beta|x-y|},
$$

therefore

$$
\big(b(x) - \operatorname{sgn}(y-x)\big) \frac{\partial}{\partial x} p_y^{+}(x,t,y) \leqslant 0
$$

as $|b|_\infty \leqslant 1$. Let $X_t$ be the diffusion process:

$$
\mathrm{d}X_t = \operatorname{sgn}(y - X_t)\,\mathrm{d}t + \mathrm{d}W_t
$$

and $\mathbb{E}_{x,y}$ denote the probability of the process $(X_s)_{0\leqslant s\leqslant t}$ conditional on $X_0 = x$ and $X_t = y$. By the Cameron-Martin formula

$$
\begin{aligned}
\frac{p_b(x,t,y)}{p_y^{+}(x,t,y)} &= \mathbb{E}_{x,y} \exp\left\{ \int_0^t \big(b(X_s) - \operatorname{sgn}(y-X_s)\big) \, \mathrm{d}W_s - \frac{1}{2}\int_0^t \big(b(X_s) - \operatorname{sgn}(y-X_s)\big)^2 \, \mathrm{d}s \right\} \\
&= \mathbb{E}_{x,y} \exp\left\{ \int_0^t \big(b(X_s) - \operatorname{sgn}(y-X_s)\big) \, \mathrm{d}\widetilde{W}_s - \frac{1}{2}\int_0^t \big(b(X_s) - \operatorname{sgn}(y-X_s)\big)^2 \, \mathrm{d}s \right. \\
&\qquad \left. + \int_0^t \big(b(X_s) - \operatorname{sgn}(y-X_s)\big) \frac{\partial}{\partial x} \log p_y^{+}(X_s, t-s, y) \, \mathrm{d}s \right\} \\
&\leqslant \mathbb{E}_{x,y} \exp\left\{ \int_0^t \big(b(X_s) - \operatorname{sgn}(y-X_s)\big) \, \mathrm{d}\widetilde{W}_s - \frac{1}{2}\int_0^t \big(b(X_s) - \operatorname{sgn}(y-X_s)\big)^2 \, \mathrm{d}s \right\} \\
&= 1,
\end{aligned}
$$

where $\widetilde{W}$ is a standard Brownian motion under the conditional probability $\mathbb{P}_{x,y}$ (see [4]) which yields that

$$
\frac{p_b(x,t,y)}{p_y^{+}(x,t,y)} \leqslant 1.
$$

Thus we have proved the upper bound in (6). Similarly, it is easy to see that

$$\big(b(x) + \operatorname{sgn}(y - x)\big)\frac{\partial}{\partial x} p_y^-(x, t, y) \geqslant 0$$

which in turn proves the lower bound in (6).

After we submitted the paper, we have learned that a quite similar formula for the density $p_y^+(x, t, z)$ has been obtained in [2] by Gradinaru, Herrmann and Roynette. In fact, in [2] a formula for $p_y^+(y, t, x)$ is given, while ours for $p_y^+(x, t, y)$, but they seem equivalent. Also we notice that there is a formula (but a little bit complicated) in ([3], formula (6.5.14)) for the density $p_y^+(x, t, z)$ under the name of diffusions with two-valued drifts. However we believe that our formula (8) is the most clean one, and our contribution thus is to bring these facts into a comparison type theorem (see Theorem 1) which we believe is useful in applications.

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