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Edge oriented reinforced random walks and RWRE

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Abstract The aim of this text is to present, in a general framework, a precise and explicit correspondance between a class of edge oriented reinforced random walks and random walks in random environment. ***To cite this article: N. Enriquez, C. Sabot, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 941–946.***

Marches aléatoires renforcées sur les arêtes orientées et marches aléatoires en milieu aléatoire

Résumé Le but de cet article est de présenter, dans un cadre général, une correspondance précise et explicite entre une classe de marches aléatoires renforcées sur les arêtes orientées et les marches aléatoires en milieu aléatoire. ***Pour citer cet article : N. Enriquez, C. Sabot, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 941–946.***

Version française abrégée

DÉFINITION 1. – On appelle loi de renforcement à d voisins, la fonction V :

$$V : \mathbb{Z}_+^d \mapsto T_d := \left\{ (x_1, \dots, x_d) \in]0,1]^d, \text{ s.t., } \sum_{i=1}^d x_i = 1 \right\},$$
$$\vec{p} = (p_1, \dots, p_d) \to \big(V_1(\vec{p}), \dots, V_d(\vec{p})\big).$$

On définit alors, pour une loi de renforcement, la notion d'admissibilité, qui jouera un rôle important dans la suite. Soit $(e_i)_{1\leqslant i\leqslant d}$ la base canonique de $\mathbb{Z}_+^d$.

DÉFINITION 2. – Considérons le graphe $\mathbb{Z}_+^d$ dont les arêtes $(\vec{p}, \vec{p}+e_i)$ sont orientées de $\vec{p}$ à $\vec{p}+e_i$. Soit V la loi de renforcement sur $\mathbb{Z}_+^d$ et w la 1-forme définie par $w((\vec{p}, \vec{p}+e_i)) = \ln V_i(\vec{p})$. On dira que V est admissible si w est une forme fermée.

On considère maintenant un graphe dénombrable G ayant en chaque point un nombre fini de voisins. Étant donné un sommet x de G, on note $d(x)$ le cardinal des voisins de x et $\{e(x,i),\ i = 1, \dots, d(x)\}$

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les voisins de x. En chaque sommet, on se donne une loi de renforcement $V(x)$ à $d(x)$ voisins : $V(x) : \vec{p} \to (V_i(x, \vec{p}))_{1 \leqslant i \leqslant d(x)}$.

DÉFINITION 3. – On appelle marche aléatoire renforcée de loi de renforcement $V(x)$, la marche aléatoire définie par la famille de loi sur les trajectoires partant de x_0, $(\widehat{P}_{x_0})_{x_0 \in G}$ donnée par

$$\widehat{P}_{x_0}\big(X_{n+1} = e(x,i) | \sigma(X_n = x) \wedge \sigma(X_k, k \leqslant n-1)\big) = V_i\big(x, \vec{N}(x)\big),$$

où $\vec{N}(x) = (N_i(x))_{1 \leqslant i \leqslant d(x)}$ et $N_i(x) = \sum_{l=0}^{n-1} 1_{\{X_l = x,\ X_{l+1} = e(x,i)\}}$.

DÉFINITION 4. – Une marche aléatoire renforcée est dite admissible si $V(x)$ est admissible pour tout sommet x de G.

Introduisons maintenant les marches aléatoires en milieu aléatoire sur le graphe G.

On définit un environnement comme un élement $\omega = (\omega(x))_{x \in G}$ où en chaque sommet x, $\omega(x)$ appartient à $T_{d(x)}$. En chaque sommet x de G, on considère une mesure de probabilité μ_x sur $T_{d(x)}$ et l'on pose $\mu := \bigotimes_{x \in G} \mu_x$. μ est une mesure de probabilité sur les environnements telle que $(\omega(x))_{x \in G}$ sont des variables aléatoires indépendantes de loi μ_x.

On note $P_{x_0,\omega}$ la loi de la chaîne de Markov dans l'environnement ω, partant de x_0, definie par :

$$\forall x_0 \in G,\ \forall k \in \mathbb{N}, \quad P_{x_0,\omega}\big(X_{k+1} = e(x,i) | X_k = x\big) = \omega(x,i).$$

Enfin, on note P_x la mesure moyennée i.e. $P_x = \mu \otimes P_{x,\omega}$. On peut alors énoncer le résultat principal :

THÉORÈME 1. – *Pour tout graphe dénombrable G,*

(i) *Pour toute loi d'environnement $\mu = \bigotimes_{x \in G} \mu_x$ la loi de la marche aléatoire renforcée $(\widehat{P}_{x_0})_{x_0 \in G}$ associée à la loi de renforcement V donnée par* :

$$V_i(x, p_1, \ldots, p_{d(x)}) = \frac{E_{\mu_x}\left[\omega(x,i) \prod_{j=1}^{d(x)} \omega(x,j)^{p_j}\right]}{E_{\mu_x}\left[\prod_{j=1}^{d(x)} \omega(x,j)^{p_j}\right]} \tag{1}$$

(où E_{μ_x} désigne l'espérance sous la loi μ_x) coincide avec la loi moyennée de la marche aléatoire en milieu aléatoire $(P_{x_0})_{x_0 \in G}$. De plus, la loi $(V(x))$ est admissible.

(ii) *Reciproquement, si V est une loi de renforcement admissible sur G, alors il existe une unique loi d'environnement $\mu = \bigotimes_{x \in G} \mu_x$ pour laquelle* (1) *est satisfaite, i.e. pour laquelle la loi de la marche aléatoire renforcée $(\widehat{P}_{x_0})_{x_0 \in G}$ coincide avec la loi moyennée $(P_{x_0})_{x_0 \in G}$.*

COROLLAIRE 1. – *La loi moyennée détermine la loi de l'environnement.*

Remarque. – On remarque, qu'une loi V admissible satisfait la propriété intuitive de renforcement, à savoir : pour tout $(i_1, \ldots, i_d) \in \mathbb{N}^d$, $V_k(i_1, \ldots, i_k + 1, \ldots, i_d) \geqslant V_k(i_1, \ldots, i_k, \ldots, i_d)$. C'est simplement une conséquence de (1) et de la positivité de la variance conditionnelle.

La preuve de (i) est directe. La preuve de (ii) s'appuye sur le fait que dans le cas d'une marche renforcée admissible, la suite des déplacements successifs depuis un sommet constitue une suite de variables échangeables. On est ensuite dans les conditions d'application du théorème de de Finetti.

1. Introduction

The aim of this text is to present in a general framework a precise and explicit correspondance between a class of edge oriented reinforced random walks and random walks in random environment (RWRE). The significance of this result is that the study of edge oriented reinforced random walks, which is strongly non-Markovian, is equivalent by introducing an external randomness, to a Markovian problem. This relation already appeared, as a tool, in a work of Pemantle (cf. [3]), for the study of reinforced random walks on binary trees, for laws of reinforcement corresponding to a Polya's urn scheme.

The essence of the result is that the laws of edge oriented reinforced random walks coincide with the annealed laws of RWRE when the laws of reinforcement satisfy the following condition: the sequence of successive moves at a given vertex is exchangeable. This condition, which we call admissiblity, is expressed as the closedness of a certain discrete form.

2. Definitions and statement of the result

DEFINITION 1. – We call law of reinforcement with d neighbours a function V:

$$V : \mathbb{Z}_+^d \mapsto T_d := \left\{ (x_1, \ldots, x_d) \in]0,1]^d, \text{ s.t., } \sum_{i=1}^d x_i = 1 \right\},$$
$$\vec{p} = (p_1, \ldots, p_d) \to \big(V_1(\vec{p}), \ldots, V_d(\vec{p})\big).$$

We now define for a law of reinforcement the notion of admissibility, which will play a key role in the following. Let $(e_i)_{1 \leqslant i \leqslant d}$ denote the canonical basis of $\mathbb{Z}_+^d$.

DEFINITION 2. – Let us consider the graph $\mathbb{Z}_+^d$ whose edges $(\vec{p}, \vec{p} + e_i)$ are oriented from $\vec{p}$ to $\vec{p} + e_i$. Let V be a law of reinforcement on $\mathbb{Z}_+^d$ and w be the 1-form defined by $w((\vec{p}, \vec{p} + e_i)) = \ln V_i(\vec{p})$. We shall say that V is admissible when w is a closed form.

We consider now a countable graph G having at any point a finite number of neighbours. For any vertex x of G we denote by $d(x)$ the cardinal of the neighbours of x and $\{e(x,i),\ i = 1, \ldots, d(x)\}$ the neighbours of x. At any vertex we suppose given a law of reinforcement $V(x)$ with $d(x)$ neighbours: $V(x) : \vec{p} \to (V_i(x, \vec{p}))_{1 \leqslant i \leqslant d(x)}$.

DEFINITION 3. – We call reinforced random walk with law of reinforcement $V(x)$, the random walk defined by the family of laws on the trajectories starting at x_0, $(\widehat{P}_{x_0})_{x_0 \in G}$ given by

$$\widehat{P}_{x_0}\big(X_{n+1} = e(x,i) | \sigma(X_n = x) \wedge \sigma(X_k, k \leqslant n-1)\big) = V_i\big(x, \vec{N}(x)\big),$$

where

$$\vec{N}(x) = \big(N_i(x)\big)_{1 \leqslant i \leqslant d(x)} \quad \text{and} \quad N_i(x) = \sum_{l=0}^{n-1} 1_{\{X_l = x, X_{l+1} = e(x,i)\}}.$$

DEFINITION 4. – A reinforced random walk is called admissible when $V(x)$ is admissible for all vertex x of G.

Let us now introduce random walks in random environment on the graph G.

We define an environment as an element $\omega = (\omega(x))_{x \in G}$ where at any vertex x, $\omega(x)$ is in $T_{d(x)}$. At any vertex x of G, we consider a probability measure μ_x on $T_{d(x)}$ and we set $\mu := \bigotimes_{x \in G} \mu_x$, so that μ is a probability measure on the environments such that $(\omega(x))_{x \in G}$ are independent random variables of law μ_x.

We denote by $P_{x_0,\omega}$ the law of the Markov chain in the environment ω starting at x_0 defined by:

$$\forall x_0 \in G,\ \forall k \in \mathbb{N}, \quad P_{x_0,\omega}\big(X_{k+1} = e(x,i) | X_k = x\big) = \omega(x,i).$$

Finally, we denote by P_x the annealed measure, i.e., $P_x = \mu \otimes P_{x,\omega}$. We are now able to state our main result:

THEOREM 1. – *For any countable graph G,*

(i) *For all law of environment $\mu=\bigotimes_{x\in G}\mu_x$ the law of the reinforced random walk $(\widehat{P}_{x_0})_{x_0\in G}$ associated with the law of reinforcement V given by*:

$$V_i(x,p_1,\ldots,p_{d(x)})=\frac{E_{\mu_x}\left[\omega(x,i)\prod_{j=1}^{d(x)}\omega(x,j)^{p_j}\right]}{E_{\mu_x}\left[\prod_{j=1}^{d(x)}\omega(x,j)^{p_j}\right]}\tag{2}$$

(where E_{μ_x} denotes the expectation under the law μ_x) coincides with the annealed law of the RWRE $(P_{x_0})_{x_0\in G}$. Moreover the law $(V(x))$ is admissible.

(ii) *Reciproquely, if V is an admissible law of reinforcement on G, then there exists a unique law of environment $\mu=\bigotimes_{x\in G}\mu_x$ for which equality* (2) *is satisfied, thus for which the law of the reinforcement random walk $(\widehat{P}_{x_0})_{x_0\in G}$ coincides with the annealed law $(P_{x_0})_{x_0\in G}$.*

COROLLARY 1. – *The annealed law determines the quenched law for RWRE, i.e., there can not exist two different laws of environment μ and μ' with the same annealed law $(P_{x_0})_{x_0\in G}$.*

Remark. – It is interesting to note that an admissible V satisfies the following intuitive property of reinforcement, i.e., for any $(i_1,\ldots,i_d)\in\mathbb{N}^d$, $V_k(i_1,\ldots,i_k+1,\ldots,i_d)\geqslant V_k(i_1,\ldots,i_k,\ldots,i_d)$. It is just a consequence of (2) and the positivity of a conditional variance.

Proof. – (i) For any vertices x_0,x of G, $\forall 1\leqslant i\leqslant d(x)$, $\forall n\in\mathbb{N}$,

$$P_{x_0}\big(X_{n+1}=e(x,i)|(X_n=x)\wedge\sigma(X_k,k\leqslant n-1)\big)=\frac{E_\mu\left[\omega(x,i)\prod_{y\in G}\prod_{j=1}^{d(y)}\omega(y,j)^{N_j(y)}\right]}{E_\mu\left[\prod_{y\in G}\prod_{j=1}^{d(y)}\omega(y,j)^{N_j(y)}\right]},$$

where $N_j(y)$ is as defined in Definition 3. Now using the independance of the variables $\omega(y,i)$ for different vertices y, the terms depending on $\omega(y,j)$ for $y\neq x$ cancel in the previous ratio and we get (i). Finally, it is straightforward from (2) that such a law of reinforcement is admissible.

(ii) The only point is to prove that for any admissible law of reinforcement V with d neigbours there exists a probability measure μ on T_d, such that a T_d-valued random variable $\vec{X}:=(X_1,\ldots,X_d)$ of law μ satisfies $V(p_1,\ldots,p_d)=E\left[X_i\prod_{j=1}^d X_j^{p_j}\right]/E\left[\prod_{j=1}^d X_j^{p_j}\right]$.

The existence for this moment problem is actually equivalent to de Finetti's theorem. Indeed, one could show that the admissibility condition we introduced implies the exchangeability of the sequence of successive moves under the natural probability measure induced by V. But for the convenience of the reader and because Corollary 1 is related to the uniqueness of the moment problem, we give an elementary proof using Haussdorff theorem.

We can see that the condition is expressed in terms of the moments of μ, therefore the aim is to prove that the assumption of admissibility on V implies the solvability of the moment problem for μ.

For that purpose we introduce the quantities which are intended to be the moments of μ.

Consider $s\in\{1,\ldots,d\}^{\mathbb{N}}$ and the path $U(s,\cdot)$ on $\mathbb{Z}_+^d$ defined by: $U(s,n):=\sum_{i=0}^{n-1}e_{s(i)}$ and let $M(s,n):=\prod_{i=0}^{n-1}V_{s(i)}(U(s,i))$.

The fact that V is admissible implies that $M(s,n)$ depends only on $U(s,n)$, indeed $M(s,n)=\exp(\int_0^{U(s,n)}w)$ (with the notations of Definition 2). So let us introduce the sequence with d indices:

$$v_{k_1,\ldots,k_d}:=M(s,n)\quad\text{for } U(s,n)=k_1e_1+\cdots+k_de_d.$$

What remains to prove is that v is the sequence of moments of a $T_{d(x)}$-valued variable. For that purpose we use the generalization of the Hausdorff criterium, concerning the existence of a (unique) solution to the moment problem for random variables on $[0,1]^d$. For any $\vec{h}=h_1,\ldots,h_d$ we define the operator $\Delta^{\vec{h}}$ on real sequences indexed by $\mathbb{Z}_+^d$, i.e., $\Delta^{\vec h} : \mathbb{R}^{\mathbb{Z}_+^d} \to \mathbb{R}^{\mathbb{Z}_+^d}$ and defined recursively by

$$\Delta^{e_i}(u) = (u_{\vec k + e_i} - u_{\vec k})_{\vec k \in \mathbb{Z}_+^d}, \quad \text{and} \quad \Delta^{\vec k + e_i} = \Delta^{e_i} \circ \Delta^{\vec k}.$$

(Remark that this definition is valid since the Δ^{e_i}'s commute.) We recall here the result of Hildebrandt and Schoenberg (cf. [2]): a sequence $(u_{\vec k}) \in [0,1]^{\mathbb{Z}_+^d}$ is the moment sequence of a probability measure on $[0,1]^d$, i.e., $u_{\vec k} = \int t_1^{k_1} \cdots t_d^{k_d} \, \mathrm{d}\mu(t_1, \dots, t_d)$ if and only if for all $\vec h$ and $\vec k$ in $\mathbb{Z}_+^d$, $(-1)^{\sum h_i} \Delta^{\vec h}(u)(\vec k)$ is positive.

Let us verify this for the sequence $v_{\vec k}$ introduced previously. Since for all $\vec k$, $\sum_{i=1}^d V_i(\vec k) = 1$ we have:

$$-\Delta^{e_i}(v)(\vec k) = \sum_{\substack{j=1 \\ j \neq i}}^{d} v_{k_1, \dots, k_j + 1, \dots, k_d}.$$

Hence, by composition $(-1)^{h_1 + \cdots + h_d} \Delta^{h_1, \dots, h_d}(v_{k_1, \dots, k_d})$ will always be a positive spanning of the terms of the sequence v. So the condition of the criterium is satisfied. Hence, μ exists and is unique as a solution of the moment problem whose support is compact. The last thing to check is that it is supported by T_d. Let $\vec X$ be a random variable with law μ. The only thing to check is that $\sum_{i=1}^d X_i = 1$ μ-almost surely. Since the law of $\sum_{i=1}^d X_i$ has compact support this is equivalent to show that all the moments of $\sum_{i=1}^d X_i$ are equal to 1. Using the fact that $\sum_{i=1}^d V_i(\vec k) = 1$ at all point $\vec k$ we know that for all integer n

$$\begin{aligned} 1 &= \sum_{s \in \{1, \dots, d\}^n} M(s, n) \\ &= \sum_{k_1 + \cdots + k_d = n} \#\bigl\{s \in \{1, \dots, d\}^n,\ U(s, n) = (k_1, \dots, k_d)\bigr\} v_{k_1, \dots, k_d} \\ &= \sum_{k_1 + \cdots + k_d = n} \frac{(k_1 + \cdots + k_d)!}{k_1! \cdots k_d!} v_{k_1, \dots, k_d}, \end{aligned}$$

and this last expression is the n-th moment of $(X_1 + \cdots + X_d)$. □

The proof of Corollary 1 is straightforward from the uniqueness of the moment problem in the case of compactly supported measure.

Example 1. – At any point x, choose a vector $(\alpha(x,1), \dots, \alpha(x, d(x)))$ in $(\mathbb{R}_+^*)^{d(x)}$. The law of reinforcement $V_i(x, p_1, \dots, p_{d(x)}) = (\alpha(x,i) + p_i) / (\sum_{j=0}^{d(x)} \alpha(x,j) + p_j)$ is an admissible law of reinforcement associated with the environment $(\mu_x)_{x \in G}$ where μ_x is a Dirichlet law with parameters $(\alpha(x,1), \dots, \alpha(x, d(x)))$, i.e., μ_x is the law on $T_{d(x)}$ with density

$$\mu_x(t_1, \dots, t_{d(x)}) = \frac{\Gamma(\alpha(x,1) + \cdots + \alpha(x, d(x)))}{\prod_{i=1}^{d(x)} \Gamma(\alpha(x,i))} \prod_{i=1}^{d(x)} t_i^{\alpha(x,i) - 1}.$$

This is the meaning of the classical Polya's urn scheme (cf. [1]), if we put at all sites an independent urn which will define the move at that site.

Example 2. – We can generalize the previous example as follows: at all point x in G, choose not only a vector $(\alpha(x,1), \dots, \alpha(x, d(x))) \in (\mathbb{R}_+^*)^{d(x)}$, but also an integer $n(x)$ and a homogeneous polynomial

$P(x,t_1,\dots,t_{d(x)})$ of degree $n(x)$ of the form:

$$P(x,t_1,\dots,t_{d(x)})=\sum_{\substack{k_1,\dots,k_{d(x)},\\ k_1+\cdots+k_{d(x)}=n(x)}} a_{k_1,\dots,k_{d(x)}}(x)t_1^{k_1}\cdots t_{d(x)}^{k_{d(x)}},$$

where the $a_{k_1,\dots,k_{d(x)}}(x)$'s are non-negative reals (not all null). Then we define μ_x as the law on $T_{d(x)}$ with density

$$\mu_x(t_1,\dots,t_{d(x)})=\frac{\left(\prod_{i=1}^{d(x)} t_i^{\alpha(x,i)-1}\right)P(x,t_1,\dots,t_{d(x)})}{\int_{T_{d(x)}}\left(\prod_{i=1}^{d(x)} t_i^{\alpha(x,i)-1}\right)P(x,t_1,\dots,t_{d(x)})}.$$

On the other side, consider the polynomials

$$Q(x,y_1,\dots,y_{d(x)})=\sum_{\substack{k_1,\dots,k_{d(x)},\\ k_1+\cdots+k_{d(x)}=n(x)}} a_{k_1,\dots,k_{d(x)}}(x)\prod_{i=1}^{d(x)}(y_i,k_i),$$

where we write (y,k) for the product $y\cdots(y+k-1)$. Then the law (μ_x) is associated with the law of reinforcement $V(x)$, where $V_i(x,p_1,\dots,p_{d(x)})$ is given by (to simplify, we forget the x dependance in the next formula, and simply write α_i for $\alpha(x,i)$ and n for $n(x)$)

$$\frac{\alpha_i+p_i}{\left(\sum_{j=1}^{d}\alpha_j+p_j\right)+n}\,\frac{Q(\alpha_1+p_1,\dots,\alpha_i+p_i+1,\dots,\alpha_d+p_d)}{Q\left(\alpha_1+p_1,\dots,\alpha_d+p_d\right)}.$$

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