C. R. Acad. Sci. Paris, Ser. I 335 (2002) 1039–1042

Géométrie algébrique/*Algebraic Geometry*

Non-reality and non-connectivity of complex polynomials

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Received 8 October 2002; accepted 15 October 2002

Note presented by Étienne Ghys.

Abstract Using the same method we provide negative answers to the following questions: is it possible to find real equations for complex polynomials in two variables up to topological equivalence (Lee Rudolph)? Can two topologically equivalent polynomials be connected by a continuous family of topologically equivalent polynomials? ***To cite this article: A. Bodin, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 1039–1042.***

Non réalité et non connectivité des polynômes complexes

Résumé Pour les polynômes de deux variables complexes, nous construisons des contre-exemples aux questions suivantes : à équivalence topologique près, peut-on toujours trouver une équation réelle à un polynôme complexe (Lee Rudolph) ? Deux polynômes topologiquement équivalents peuvent-ils être reliés par une famille de polynômes topologiquement équivalents ? ***Pour citer cet article : A. Bodin, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 1039–1042.***

1. Introduction

Two polynomials $f, g \in \mathbb{C}[x, y]$ are *topologically equivalent*, and we will denote $f \approx g$, if there exist homeomorphisms $\Phi : \mathbb{C}^2 \to \mathbb{C}^2$ and $\Psi : \mathbb{C} \to \mathbb{C}$ such that $g \circ \Phi = \Psi \circ f$. They are *algebraically equivalent*, and we will denote $f \sim g$, if we have $\Phi \in \operatorname{Aut}\mathbb{C}^2$ and $\Psi = \mathrm{id}$.

It is always possible to find real equations for germs of plane curves up to topological equivalence. In fact the proof is as follows: the topological type of a germ of plane curve $(C, 0)$ is determined by the characteristic pairs of the Puiseux expansions of the irreducible branches and by the intersection multiplicities between these branches. Then we can choose the coefficients of the Puiseux expansions in $\mathbb{R}$ (even in $\mathbb{Z}$). Now it is possible (see [7], appendix to Chapter 1) to find a polynomial $f \in \mathbb{R}[x, y]$ (even in $\mathbb{Z}[x, y]$) such that the germ $(f = 0, 0)$ is equivalent to the germ $(C, 0)$.

This property has been widely used by N. A'Campo and others (see [1] for example) in the theory of divides. Lee Rudolph asked the question whether it is true for polynomials [10]. We give a negative answer:

THEOREM A. – *Up to topological equivalence it is not always possible to find real equations for complex polynomials.*

2. We now deal with another problem. In [5] we proved that a family of polynomials with some constant numerical data are all topologically equivalent. More precisely for a polynomial let $\mathfrak{m} = (\mu, \#\mathcal{B}_{\mathrm{aff}}, \lambda, \#\mathcal{B}_\infty, \#\mathcal{B})$ be the multi-integer respectively composed of the affine Milnor number, the number

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S1631-073X(02)02597-9/FLA

of affine critical values, the Milnor number at infinity, the number of critical values at infinity, the number of critical values (with $\mathcal{B} = \mathcal{B}_{\text{aff}} \cup \mathcal{B}_\infty$). Then we have a global version of the Lê-Ramanujam μ-constant theorem:

THEOREM ([5]). – *Let $(f_t)_{t\in[0,1]}$ be a family of complex polynomials in two variables whose coefficients are polynomials in t. Suppose that the multi-integer $\mathfrak{m}(t)$ and the degree $\deg f_t$ do not depend on $t \in [0,1]$. Then the polynomials f_0 and f_1 are topologically equivalent.*

It is true that two topologically equivalent polynomials have the same multi-integers $\mathfrak{m}$. A natural question is: can two topologically equivalent polynomials be connected by a continuous family of topologically equivalent polynomials?

THEOREM B. – *There exist two topologically equivalent polynomials f_0, f_1 that cannot be connected by a family of equivalent polynomials. That means that for each continuous family $(f_t)_{t\in[0,1]}$ there exists a $\tau \in]0,1[$ such that f_τ is not topologically equivalent to f_0.*

It can be noticed that the answer is positive for algebraic equivalence. Two algebraically equivalent polynomials can be connected by algebraically equivalent polynomials since $\operatorname{Aut}\mathbb{C}^2$ is connected by Jung's theorem.

Such kinds of problems have been studied by V. Kharlamov and V. Kulikov in [9] for cuspidal projective curves. They give two complex conjugate projective curves that are not isotopic. The example with lowest degree has degree 825. In [2], Artal, Carmona and Cogolludo give examples of projective curves C, C' of degree 6 that have conjugate equations in $\mathbb{Q}(\sqrt{2})$ but the pairs $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C')$ are not homeomorphic by an orientation-preserving homeomorphism.

3. The method used in this note is based on the relationship between topological and algebraic equivalence: we set a family $(f_s)_{s\in\mathbb{C}}$ of polynomials such that $(f_s = 0)$ is a line arrangement in $\mathbb{C}^2$. One of the line depends on a parameter $s \in \mathbb{C}$. There are enough lines in order that each polynomial is algebraically essentially unique. Moreover every polynomial topologically equivalent to f_s is algebraic equivalent to a $f_{s'}$, where s' may be different from s.

For generic parameters the polynomials are topologically equivalent all together and the function f_s is a Morse function on $\mathbb{C}^2 \setminus f_s^{-1}(0)$. We choose our counter-examples with non-generic parameters, for such an example f_k is not a Morse function on $\mathbb{C}^2 \setminus f_k^{-1}(0)$. The fact that non-generic parameters are finite enables us to prove the requested properties.

4. Non-reality

Let

$$f_s(x,y) = xy(x-y)(y-1)(x-sy).$$

Let $k, \bar{k}$ be the roots of $s^2 - s + 1$.

THEOREM A. – *There does not exist a polynomial g with real coefficients such that $g \approx f_k$.*

Let $\mathcal{C} = \{0, 1, k, \bar{k}\}$. Then for $s \in \mathbb{C} \setminus \mathcal{C}$, f_s verifies $\mu = 14$, $\#\mathcal{B}_{\text{aff}} = 3$ and $\mathcal{B}_\infty = \varnothing$. By the connectivity of $\mathbb{C} \setminus \mathcal{C}$ and the global version of the μ-constant theorem, two polynomials f_s and $f_{s'}$, with $s, s' \notin \mathcal{C}$, are topologically equivalent.

The polynomials f_k and $f_{\bar{k}}$ verify $\mu = 14$, but $\#\mathcal{B}_{\text{aff}} = 2$. Then such a polynomial is not topologically equivalent to a generic one f_s, $s \notin \mathcal{C}$. In fact for $s \notin \mathcal{C}$ there are two non-zero critical fibers with one double point for each one. For $s = k$ or $s = \bar{k}$, there is only one non-zero critical fiber with an ordinary cusp.

LEMMA 1. – *Let $s, s' \in \mathbb{C}$. The polynomials f_s and $f_{s'}$ are algebraically equivalent if and only if $s = s'$ or $s = 1 - s'$.*

In particular the polynomials f_k and $f_{\bar{k}}$ are algebraically equivalent.

Proof. – Let us suppose that f_s and $f_{s'}$ are algebraically equivalent. Then we can suppose that there exists $\Phi \in \operatorname{Aut}\mathbb{C}^2$ such that $f_{s'} = f_s \circ \Phi$. Such a Φ must send the lines $(x = 0)$, $(y = 0)$ to two lines, then Φ is linear: $\Phi(x, y) = (ax + by, cx + dy)$. A calculus proves that $\Phi(x, y) = (x, y)$ or $\Phi(x, y) = (y - x, y)$ that is to say $s = s'$ or $s = 1 - s'$. □

LEMMA 2. – *Fix $s \in \mathbb{C}$ and let f be a polynomial such that $f \approx f_s$. There exists s' such that $f \sim f_{s'}$.*

Then Lemma 1 implies that there are only two choices for s', but s' can be different from s.

Proof. – The curve $f_s^{-1}(0)$ contains the simply connected curve $xy(x - y)(x - sy)$, then the curve $f^{-1}(0)$ contains also a simply connected curve (with 4 components), by the generalization of Zaïdenberg–Lin theorem (see [4]) this simply connected curve is algebraically equivalent to $xy(x - y)(x - s'y)$. Then the polynomial f is algebraically equivalent to $xy(x - y)(x - s'y)P(x, y)$. The curve C defined by $(P = 0)$ is homeomorphic to $\mathbb{C}$ and admits a polynomial parameterization $(\alpha(t), \beta(t))$ with $\alpha, \beta \in \mathbb{C}[t]$. Since C does not intersect the axe $(y = 0)$, β is a constant polynomial; since C intersects the axe $(x = 0)$ at one point α is monomial. An equation of P is now $P(x, y) = y^n - \lambda$. By the irreducibility of C and up to an homothety we get $P(x, y) = y - 1$. That is to say f is algebraically equivalent to $f_{s'}$. □

5. Let $g \in \mathbb{C}[x, y]$, if $g(x, y) = \sum a_{i,j} x^i y^j$ then we denote by $\bar{g}$ the polynomial defined by $\bar{g}(x, y) = \sum \bar{a}_{i,j} x^i y^j$. Then $g = \bar{g}$ if and only if all the coefficients of g are real.

We prove Theorem A. Let suppose that there exists a polynomial g such that $g = \bar{g}$ and $g \approx f_k$. There exists $s \in \mathbb{C}$ such that $g \sim f_s$. Since f_k has only two critical values, g and f_s have two critical values. Then $s = k$ or $s = \bar{k}$ ($s = 0$ or $s = 1$ gives a polynomial with non-isolated singularities). As $f_k \sim f_{\bar{k}}$ we can choose $s = k$. As a consequence we have $\Phi \in \operatorname{Aut}\mathbb{C}^2$ such that $g = f_k \circ \Phi$.

Let Φ be $\Phi = (p, q)$. Then $g = pq(p - q)(q - 1)(p - kq)$. As $g = \bar{g}$ we have:

$$\{p, q, p - q, q - 1, p - kq\} = \{\bar{p}, \bar{q}, \bar{p} - \bar{q}, \bar{q} - 1, \bar{p} - \bar{k}\bar{q}\}.$$

Moreover by the configuration of the lines we have that $q - 1 = \bar{q} - 1$. So $q = \bar{q}$. Hence $q \in \mathbb{R}[x, y]$. So

$$\{p, p - q, p - kq\} = \{\bar{p}, \bar{p} - \bar{q}, \bar{p} - \bar{k}\bar{q}\}.$$

Let suppose that $p \neq \bar{p}$. Then $p = \bar{p} - q$ or $p = \bar{p} - \bar{k}q$. So $p - \bar{p}$ equals $-q$ or $-\bar{k}q$. But $p - \bar{p}$ has coefficients in $\mathrm{i}\mathbb{R}$, which is not the case of $q \in \mathbb{R}[x, y]$ nor of $\bar{k}q$. Then $p = \bar{p}$. We have proved that $\Phi = (p, q)$ has real coefficients. From $g = f_k \circ \Phi$ we get $\bar{g} = \bar{f_k} \circ \bar{\Phi}$. So $g = f_{\bar{k}} \circ \Phi$. On the one hand $f_k = g \circ \Phi^{-1}$ and on the other hand $f_{\bar{k}} = g \circ \Phi^{-1}$. So $f_k = f_{\bar{k}}$, then $k = \bar{k}$ which is false. It ends the proof.

We could have end in the following way: $\Phi = (p, q)$ is in $\operatorname{Aut}\mathbb{C}^2$ with real coefficients, then Φ, considered as a real map, is in $\operatorname{Aut}\mathbb{R}^2$ (see [3, Theorem 2.1] for example). Then $f_k = g \circ \Phi^{-1}$ with g, Φ^{-1} with real coefficients, then f_k has real coefficients which provides the contradiction.

6. Non-connectivity

Let

$$f_s(x, y) = xy(y - 1)(x + y - 1)(x - sy).$$

Let $\mathcal{C}$ be the roots of

$$s(s - 1)(s + 1)(256s^4 + 736s^3 + 825s^2 + 736s + 256)(256s^4 + 448s^3 + 789s^2 + 448s + 256).$$

Then for $s \in \mathbb{C} \setminus \mathcal{C}$, f_s verifies $\mu = 14$, $\#\mathcal{B}_{\mathrm{aff}} = 4$ and $\mathcal{B}_\infty = \varnothing$. For $s, s' \notin \mathcal{C}$, f_s and $f_{s'}$ are topologically equivalent. The roots of $256s^4 + 448s^3 + 789s^2 + 448s + 256$ are of the form $\{k, \bar{k}, 1/k, 1/\bar{k}\}$. The polynomials f_k and $f_{\bar{k}}$ verify $\mu = 14$, but $\#\mathcal{B}_{\mathrm{aff}} = 3$. Then such a polynomial is not topologically equivalent to a generic one f_s, $s \notin \mathcal{C}$.

THEOREM B. – *The polynomials f_k and $f_{\bar{k}}$ are topologically equivalent and it is not possible to find a continuous family $(g_t)_{t\in[0,1]}$ such that $g_0 = f_k$, $g_1 = f_{\bar{k}}$ and $g_t \approx f_k$ for all $t \in [0,1]$.*

The polynomials f_k and $f_{\bar{k}}$ are topologically equivalent since we have the formula $f_{\bar{k}}(\bar{x}, \bar{y}) = \overline{f_k(x, y)}$. The two following lemmas are similar to Lemmas 1 and 2.

LEMMA 3. – *The polynomials f_s and $f_{s'}$ are algebraically equivalent if and only if $s = s'$ or $s = 1/s'$.*

LEMMA 4. – *Fix s and let f be a polynomial such that $f \approx f_s$. Then there exists s' such that $f \sim f_{s'}$.*

7. We now prove Theorem B. Let us suppose that such a family (g_t) does exist. Then by Lemma 4 for each $t \in [0,1]$ there exists $s(t) \in \mathbb{C}$ such that g_t is algebraically equivalent to $f_{s(t)}$ (in fact there are two choices for $s(t)$). We can suppose that there exists $\Phi_t \in \operatorname{Aut}\mathbb{C}^2$ such that $f_{s(t)} = g_t \circ \Phi_t$.

We now prove that the map $t \mapsto \Phi_t$ can be chosen continuous, that is to say the coefficients of the defining polynomials are continuous functions of t. We write $g_t = A_t B_t G_t$ such that $A_0(x, y) = x$, $B_0(x, y) = y$ and the maps $t \mapsto A_t$, $t \mapsto B_t$ are continuous. So the automorphism Φ_t^{-1} is defined by

$$\Phi_t^{-1}(x, y) = \big(A_t(x, y), B_t(x, y)\big).$$

By the inverse local theorem with parameter t, we have that $t \mapsto \Phi_t$ is a continuous function. Then the map $t \mapsto f_{s(t)}$ is a continuous function, as the composition of two continuous functions. As $s(t)$ is a coefficient of the polynomial $f_{s(t)}$, the map $t \mapsto s(t)$ is a continuous function.

As a conclusion we have a map $t \mapsto s(t)$ which is continuous and such that $s(0) = k$ and $s(1) = \bar{k}$. It implies that there exists $\tau \in]0,1[$ such that $s(\tau) \notin \mathcal{C}$. On the one hand g_τ is algebraically, hence topologically, equivalent to $f_{s(\tau)}$; on the other hand g_τ is topologically equivalent to f_k (by hypothesis). As $s(\tau) \notin \mathcal{C}$, $f_{s(\tau)}$ and f_k are not topologically equivalent (because $\#\mathcal{B}_{\text{aff}}$ are different), it provides a contradiction.

8. I would like to thank Lee Rudolph for the question which initiated this work. The calculus have been done with the help of SINGULAR, [8], and especially with author's library `critic` described in [6]. This research has been done at the *Centre de Recerca Matemàtica* of Barcelona and was supported by a Marie Curie Individual Fellowship of the European Community (HPMF-CT-2001-01246).

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