

Nondegeneracy of the Lie algebra $\mathfrak{aff}(n)$

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Abstract

We show that $\mathfrak{aff}(n)$, the Lie algebra of affine transformations of \mathbb{R}^n , is formally and analytically nondegenerate in the sense of A. Weinstein. This means that every analytic (resp., formal) Poisson structure vanishing at a point with a linear part corresponding to $\mathfrak{aff}(n)$ is locally analytically (resp., formally) linearizable. *To cite this article: J.-P. Dufour, N.T. Zung, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 1043–1046.*

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Non-dégénérescence de l'algèbre de Lie $\mathfrak{aff}(n)$

Résumé

Nous montrons que toute structure de Poisson analytique (resp., formelle), qui s'annule en un point et dont la partie linéaire correspond à l'algèbre $\mathfrak{aff}(n)$ des transformations affines sur \mathbb{R}^n , est localement analytiquement (resp., formellement) linéarisable. *Pour citer cet article : J.-P. Dufour, N.T. Zung, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 1043–1046.*

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1. Introduction

Following Weinstein [9], we will say that a Lie algebra \mathfrak{g} is formally (resp. analytically, smoothly) nondegenerate if any formal (resp. analytic, smooth) Poisson structure Π vanishing at a point, with a linear part $\Pi^{(1)}$ corresponding to \mathfrak{g} , is formally (resp. analytically, smoothly) linearizable. An interesting and largely open question in Poisson geometry is to find and classify nondegenerate Lie algebras in the above senses. Up to now, only few nondegenerate Lie algebras are known. These include: the nontrivial 2-dimensional Lie algebra $\mathfrak{aff}(1)$ (see appendix of [1] on Poisson structures and densities); semisimple Lie algebras [9,2,3]; a complete list of nondegenerate Lie algebras in dimensions 3 [4] and 4 [6], direct products of semisimple algebras with \mathbb{R} (or \mathbb{C}) [6], and direct products of n copies of $\mathfrak{aff}(1)$ [5].

Recent works by A. Wade, Ph. Monnier and the second author on Levi decomposition of Poisson structures [7,8,10] open a new way of linearizing Poisson structures by first looking for a semi-linearization associated to a Levi decomposition of their linear part. We will recall this method in Section 2. Using it, we obtain the following result:

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THEOREM 1.1. – *For any natural number n , the Lie algebra $\mathfrak{aff}(n, \mathbb{K}) = \mathfrak{gl}(n, \mathbb{K}) \ltimes \mathbb{K}^n$ of affine transformations of \mathbb{K}^n , where $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , is formally and analytically nondegenerate.*

The above theorem provides a new entry in a very short list of known examples of nondegenerate Lie algebras. Moreover unlike previously known examples, $\mathfrak{aff}(n)$ for $n \geq 2$ is neither reductive nor solvable. We remark that the main sisters of $\mathfrak{aff}(n)$, namely the algebra $\mathfrak{e}(n) = \mathfrak{so}(n) \ltimes \mathbb{R}^n$ of Euclidean transformations and the algebra $\mathfrak{saff}(n) = \mathfrak{sl}(n) \ltimes \mathbb{K}^n$ of volume preserving affine transformations, are unfortunately degenerate for $n = 2$ or 3 , as simple polynomial non-linearizable examples show (we suspect that they are degenerate for $n > 2$ as well).

Another interesting feature about $\mathfrak{aff}(n)$ is that it is a Frobenius Lie algebra, in the sense that there is a dense open subset in the dual of $\mathfrak{aff}(n)$ where the corresponding linear Poisson structure is nondegenerate. The existence of an open coadjoint orbit probably plays a role in the nondegeneracy of $\mathfrak{aff}(n)$ and some other Frobenius Lie algebras. However, being Frobenius does not guarantee nondegeneracy: already in dimension 4 there are counter-examples, which can be seen from the list given in [6].

The rest of this Note is organized as follows: in Section 2 we obtain an improved version of the analytic semi-linearization result of [10], which works not only for $\mathfrak{aff}(n)$ but for many other Lie algebras as well. Then in Section 3 we give a proof of Theorem 1.1 based on this improved semi-linearization and using a trick involving Casimir functions for $\mathfrak{gl}(n)$. For simplicity of exposition, we will restrict our attention to the analytic case. The formal case is absolutely similar, if not simpler. In Section 4 we show that the Lie algebras $\mathfrak{saff}(2)$ and $\mathfrak{e}(3)$ are degenerate (formally, analytically and smoothly).

2. Semi-linearization for $\mathfrak{aff}(n)$

Denote by $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$ a Levi decomposition for a (real or complex) Lie algebra \mathfrak{g} , where \mathfrak{s} is semisimple and \mathfrak{r} is the solvable radical. Let Π be an analytic Poisson structure vanishing at a point 0 in a manifold whose linear part at 0 corresponds to \mathfrak{g} . According to the main result of [10] (called the analytic Levi decomposition theorem), there exists a local analytic system of coordinates $(x_1, \dots, x_m, y_1, \dots, y_d)$ in a neighborhood of 0, where $m = \dim \mathfrak{s}$ and $d = \dim \mathfrak{r}$, such that in these coordinates we have

$$\{x_i, x_j\} = \sum c_{ij}^k x_k, \quad \{x_i, y_r\} = \sum a_{ir}^s y^s, \tag{1}$$

where c_{ij}^k are structural constants of \mathfrak{s} and a_{ir}^s are constants. This gives what we call a semi-linearization for Π . Note that the remaining Poisson brackets $\{y_r, y_s\}$ are nonlinear in general.

We now restrict our attention to the case where $\mathfrak{g} = \mathfrak{aff}(n)$, $m = n^2 - 1$, $d = n + 1$, $\mathfrak{s} = \mathfrak{sl}(n)$, $\mathfrak{r} = \mathbb{K}(\text{Id}) \ltimes \mathbb{K}^n$ where Id acts on \mathbb{K}^n by the identity map. The following lemma says that we may have a semi-linearization associated to the decomposition $\mathfrak{aff}(n) = \mathfrak{gl}(n) \ltimes \mathbb{K}^n$ (which is better than the Levi decomposition).

LEMMA 2.1. – *There are local analytic coordinates $x_1, \dots, x_{n^2-1}, y_0, y_1, \dots, y_n$ which satisfy relations (1), with the following extra properties: $\{y_0, y_r\} = y_r$ for $r = 1, \dots, n$; $\{x_i, y_0\} = 0 \forall i$.*

Proof. – We can assume that the coordinates y_r are chosen so that relations (1) are already satisfied, and y_0 corresponds to Id in $\mathbb{K}(\text{Id}) \ltimes \mathbb{K}^n$. Then the Hamiltonian vector fields X_{x_i} are linear and form a linear action of $\mathfrak{sl}(n)$. Because of (1), we have that $\{x_i, y_0\} = 0$, which implies that $[X_{x_i}, X_{y_0}] = 0$, i.e., X_{y_0} is invariant under the $\mathfrak{sl}(n)$ action. Moreover we have $X_{y_0}(x_i) = 0$ (i.e., X_{y_0} does not contain components $\partial/\partial x_i$), and $X_{y_0} = \sum_1^n y_i \partial/\partial y_i +$ nonlinear terms. Hence we can use (the parametrized equivariant version of) Poincaré linearization theorem to linearize X_{y_0} in a $\mathfrak{sl}(n)$ -invariant way. After this linearization, we have that $X_{y_0} = \sum_1^n y_i \partial/\partial y_i$. In other words, relations (1) are still satisfied, and moreover we have $\{y_0, y_i\} = X_{y_0}(y_i) = y_i$. \square

The significance of the above lemma is that we can extend the analytic semi-linearization of [10] from $\mathfrak{sl}(n)$ to $\mathfrak{gl}(n) = \mathfrak{sl}(n) \oplus \mathbb{K}$. Hereafter we will redenote y_0 in the above lemma by x_{n^2} . Then relations (1) are still satisfied.

Remark. – Lemma 2.1 still holds if we replace $\text{aff}(n)$ by any Lie algebra of the type $(\mathfrak{s} \oplus \mathbb{K}e_0) \ltimes \mathfrak{n}$ where \mathfrak{s} is semisimple and e_0 acts on \mathfrak{n} by the identity map (or any matrix whose corresponding linear vector field is nonresonant and satisfies a Diophantine condition).

3. Linearization for $\text{aff}(n)$

We will work in a coordinate system $x_1, \dots, x_{n^2}, y_1, \dots, y_n$ provided by Lemma 2.1. We will fix the variables x_1, \dots, x_{n^2} , and consider them as linear functions on $\mathfrak{gl}^*(n)$ (they give a Poisson projection from our $(n^2 + n)$ -dimensional space to $\mathfrak{gl}^*(n)$). Denote by F_1, \dots, F_n the n basic Casimir functions for $\mathfrak{gl}^*(n)$ (if we identify $\mathfrak{gl}(n)$ with its dual via the Killing form, then F_1, \dots, F_n are basic symmetric functions of the eigenvalues of $n \times n$ matrices). We will consider $F_1(x), \dots, F_n(x)$ as functions in our $(n^2 + n)$ -dimensional space, which do not depend on variables y_i . Denote by X_1, \dots, X_n the Hamiltonian vector fields of F_1, \dots, F_n .

LEMMA 3.1. – *The vector fields X_1, \dots, X_n do not contain components $\partial/\partial x_i$. They form a system of n linear commuting vector fields on \mathbb{K}^n (the space of $y = (y_1, \dots, y_n)$) with coefficients which are polynomial in $x = (x_1, \dots, x_{n^2})$. The set of x such that they are linearly dependent everywhere in \mathbb{K}^n is an analytic space of complex codimension greater than 1 (when $\mathbb{K} = \mathbb{C}$).*

Proof. – The fact that the X_i are y -linear with x -polynomial coefficients follows directly from relations (1). Since F_i are Casimir functions for $\mathfrak{gl}(n)$, we have $X_i(x_k) = \{F_i, x_k\} = 0$, and $[X_i, X_j] = X_{\{F_i, F_j\}} = 0$.

One checks that, for a given x , $X_1 \wedge \dots \wedge X_n = 0$ identically on \mathbb{K}^n if and only if x is a singular point for the map (F_1, \dots, F_n) from $\mathfrak{gl}^*(n)$ to \mathbb{K}^n . The set of singular points of the map (F_1, \dots, F_n) in the complex case is of codimension greater than 1 (in fact, it is of codimension 3). \square

LEMMA 3.2. – *Write the Poisson structure Π in the form $\Pi = \Pi^{(1)} + \tilde{\Pi}$, where $\Pi^{(1)}$ is the linear part and $\tilde{\Pi}$ denote the higher order terms. Then $\tilde{\Pi}$ is a Poisson structure which can be written in the form*

$$\tilde{\Pi} = \sum_{i < j} f_{ij} X_i \wedge X_j, \tag{2}$$

where the functions f_{ij} are analytic functions which depend only on the variables x , and they are Casimir functions for $\mathfrak{gl}^*(n)$ (if we consider the variables x as linear functions on $\mathfrak{gl}^*(n)$).

Proof. – We work first locally near a point (x, y) where the vector fields X_k are linearly independent point-wise. As $\tilde{\Pi}$ is a 2-vector field in $\mathbb{K}^n = \{y\}$ (with coefficient depending on x) we have a local formula $\tilde{\Pi} = \sum_{i < j} f_{ij} X_i \wedge X_j$ where f_{ij} are analytic functions in variables (x, y) . Since X_k are Hamiltonian vector fields for Π and also for $\Pi^{(1)}$, we have $[X_k, \tilde{\Pi}] = [X_k, \Pi] - [X_k, \Pi^{(1)}] = 0$ for $k = 1, \dots, n$. This leads to $X_k(f_{ij}) = 0 \forall k, i, j$. Hence, because the X_k generate \mathbb{K}^n , the functions f_{ij} are locally independent of y . Using analytic extension, Hartog’s theorem and the fact that the set of x such that X_1, \dots, X_n are linearly independent point-wise everywhere in \mathbb{K}^n is of complex codimension greater than 1, we obtain that f_{ij} are local analytic functions in a neighborhood of 0 which depend only on the variables x . The fact that $\tilde{\Pi}$ is a Poisson structure, i.e., $[\tilde{\Pi}, \tilde{\Pi}] = 0$, is now evident, because $X_k(f_{ij}) = 0$ and $[X_i, X_j] = 0$.

Relations $[X_{x_k}, \tilde{\Pi}] = [X_{x_i}, \Pi] - [X_{x_i}, \Pi^{(1)}] = 0$ imply that $X_{x_k}(f_{ij}) = 0$, which means that f_{ij} are Casimir functions for $\mathfrak{gl}^*(n)$. \square

Remark. – Lemma 3.2 is still valid in the formal case. In fact, every homogeneous component of $\tilde{\Pi}$ satisfies a relation of type (2).

LEMMA 3.3. – *There exists a vector field Y of the form $Y = \sum_{i=1}^n \alpha_i X_i$, where the analytic functions α_i depend only on the variables x and are Casimir functions for $\mathfrak{gl}^*(n)$, such that*

$$[Y, \Pi^{(1)}] = -\tilde{\Pi}, \quad [Y, \tilde{\Pi}] = 0. \tag{3}$$

Proof. – Since the functions f_{ij} of Lemma 3.2 are analytic Casimir functions for $\mathfrak{gl}(n)$, we have $f_{ij} = \phi_{ij}(F_1, \dots, F_n)$ where $\phi_{ij}(z_1, \dots, z_n)$ are analytic functions of n variables. On the other hand, since $\Pi^{(1)}$, $\tilde{\Pi}$ and $\Pi = \Pi^{(1)} + \tilde{\Pi}$ are Poisson structures, they are compatible, i.e., we have $[\Pi^{(1)}, \tilde{\Pi}] = 0$. Decomposing this relation, we get $\frac{\partial \phi_{ij}}{\partial z_k} + \frac{\partial \phi_{jk}}{\partial z_i} + \frac{\partial \phi_{ki}}{\partial z_j} = 0 \forall i, j, k$. This is equivalent to the fact that the 2-form $\phi := \sum_{ij} \phi_{ij} dz_i \wedge dz_j$ is closed. By Poincaré’s lemma we get $\phi = d\alpha$ with an 1-form $\alpha = \sum_i \alpha_i dz_i$. Then we put $Y := \sum_i \alpha_i (F_1, \dots, F_n) X_i$. An elementary calculation proves that Y is the desired vector field. \square

Proof of Theorem 1.1. – Consider a path of Poisson structures given by $\Pi_t := \Pi^{(1)} + t\tilde{\Pi}$. As we have $[Y, \Pi_t] = \tilde{\Pi} = \frac{d}{dt} \Pi_t$, the time-1 map of the vector field Y moves $\Pi^{(1)} = \Pi_0$ into $\Pi = \Pi_1$. This shows that Π is locally analytically linearizable, thus proving our theorem. \square

4. Degeneracy of $\mathfrak{sa}ff(2)$ and $\epsilon(3)$

The linear Poisson structure corresponding to $\mathfrak{sa}ff(2)$ has the form $\Pi^{(1)} = 2e\partial h \wedge \partial e - 2f\partial h \wedge \partial f + h\partial e \wedge \partial f + y_1\partial h \wedge \partial y_1 - y_2\partial h \wedge \partial y_2 + y_1\partial e \wedge \partial y_2 + y_2\partial f \wedge \partial y_1$ in a natural system of coordinates. Now put $\Pi = \Pi^{(1)} + \tilde{\Pi}$ with $\tilde{\Pi} = (h^2 + 4ef)\partial y_1 \wedge \partial y_2$. Then Π is a Poisson structure, vanishing at the origin, with a linear part corresponding to $\mathfrak{sa}ff(2)$. For $\Pi^{(1)}$ the set where the rank is less or equal to 2 is a codimension 2 subspace (given by the equations $y_1 = 0$ and $y_2 = 0$). For Π the set where the rank is less or equal to 2 is a 2-dimensional cone (the cone given by the equations $y_1 = 0, y_2 = 0$ and $h^2 + 4ef = 0$). So these two Poisson structures are not isomorphic, even formally.

The linear Poisson structure corresponding to $\epsilon(3)$ has the form $\Pi^{(1)} = x_1\partial x_2 \wedge \partial x_3 + x_2\partial x_3 \wedge \partial x_1 + x_3\partial x_1 \wedge \partial x_2 + y_1\partial x_2 \wedge \partial y_3 + y_2\partial x_3 \wedge \partial y_1 + y_3\partial x_1 \wedge \partial y_2$ in a natural system of coordinates. Now put $\Pi = \Pi^{(1)} + \tilde{\Pi}$ with $\tilde{\Pi} = (x_1^2 + x_2^2 + x_3^2)(x_1\partial y_2 \wedge \partial y_3 + x_2\partial y_3 \wedge \partial y_1 + x_3\partial y_1 \wedge \partial y_2)$. For $\Pi^{(1)}$ the set where the rank is less or equal to 2 is a dimension 3 subspace (given by the equation $y_1 = y_2 = y_3 = 0$), while for Π the set where the rank is less or equal to 2 is the origin.

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