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Lower bounds for the counting function of resonances for a perturbation of a periodic Schrödinger operator by decreasing potential

Mouez Dimassi [a], Maher Mnif [b]

[a] Institut Galilée, avenue Jean-Baptiste-Clément, 93430 Villetaneuse, France
[b] I.P.E.I.S., boite postale 805, Sfax 3000, Tunisie

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Abstract We are interested here in the counting function of resonances $N(h)$ for a perturbation of a periodic Schrödinger operator P_0 by decreasing potential $W(hx)$ ($h \searrow 0$). We obtain a lower bound for $N(h)$ near some singularities of the density of states measure, associated to the unperturbed Hamiltonian P_0. ***To cite this article: M. Dimassi, M. Mnif, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 1013–1016.***

Des minorations de la fonction de comptage de résonances pour une perturbation d'un opérateur de Schrödinger périodique par un potientiel décroissant

Résumé On s'intéresse ici à la fonction de comptage $N(h)$ du nombre de résonances de l'opérateur de Schrödinger périodique P_0 perturbé par un potentiel décroissant $W(hx)$ ($h \searrow 0$). Nous obtenons une minoration de $N(h)$ près de certaines singularités de la densité d'états associée à l'opérateur non perturbé P_0. ***Pour citer cet article : M. Dimassi, M. Mnif, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 1013–1016.***

1. Introduction

The purpose of this paper is to give a lower bound for the counting function of resonances for the perturbed periodic Schrödinger operator:

$$P(h) = P_0 + W(hx), \quad P_0 = -\Delta + V(x)\ (h \searrow 0).$$

Here V is C^∞, real-valued and Γ-periodic with respect to a lattice $\Gamma = \bigoplus_{i=1}^n \mathbf{Z}e_i$ in $\mathbf{R}^n$. The potential W is real-valued and satisfies:

E-mail addresses: dimassi@math.univ-paris13.fr (M. Dimassi); maher.mnif@ipeis.rnu.tn (M. Mnif).

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(H1) there exist positive constants a and C such that W extends analytically to $\Gamma(a) := \{z \in \mathbf{C}^n;\ |\Im(z)| \leqslant a\langle \Re(z)\rangle\}$ and

$$\bigl|W(z)\bigr| \leqslant C\langle z\rangle^{-\tilde{n}}, \quad \text{uniformly on } z \in \Gamma(a), \tilde{n} > n, \tag{1}$$

where $\langle z\rangle = (1+|z|^2)^{1/2}$. Here $\Re(z)$, $\Im(z)$ denote respectively the real part and the imaginary part of z.

For $k \in \mathbf{R}^n$, we define the operator P_k on $\mathrm{L}^2(\mathbf{R}^n/\Gamma)$ by:

$$P_k := (D_y + k)^2 + V(y).$$

The Floquet eigenvalues are the eigenvalues $\lambda_1(k) \leqslant \lambda_2(k) \leqslant \cdots$ of P_k (enumerated according to their multiplicities). It is well known that [3]:

$$\sigma(P_0) = \sigma_{\mathrm{ac}}(P_0) = \bigcup_{j\geqslant 1} \Lambda_j, \quad \Lambda_j = \lambda_j\bigl(\mathbf{R}^n/\Gamma^*\bigr).$$

Here Γ^* is the dual lattice corresponding to Γ.

For $f \in \mathrm{C}_0^\infty(\mathbf{R})$, we set

$$\langle \mu, f\rangle = \int \bigl[f\bigl(W(x)\bigr) - f(0)\bigr]\,\mathrm{d}x, \tag{2}$$

$$\langle \omega, f\rangle = \sum_{j\geqslant 1} \int_{E^*}\int_{\mathbf{R}_x^n} \bigl[f\bigl(W(x) + \lambda_j(k)\bigr) - f\bigl(\lambda_j(k)\bigr)\bigr]\,\mathrm{d}k\,\mathrm{d}x, \tag{3}$$

where E^* is a fundamental domain of $\mathbf{R}^n/\Gamma^*$.

PROPOSITION 1. – *The functionals operators ω and μ are distributions on $\mathbf{R}$ of order $\leqslant 1$. Moreover, in $\mathcal{D}'(\mathbf{R})$, we have*

$$\omega = \mathrm{d}\rho * \mu. \tag{4}$$

Here

$$\rho(\lambda) := \frac{1}{(2\pi)^n} \sum_{j\geqslant 1} \int_{\{k\in E^*;\ \lambda_j(k)\leqslant\lambda\}} \mathrm{d}k, \tag{5}$$

is the density of states measure associated to the unperturbed Hamiltonian P_0.

Proof. – Applying Taylor's formula to the r.h.s. of (2), we obtain

$$\bigl|\langle \mu, f\rangle\bigr| \leqslant \sup|f'| \int \bigl|W(x)\bigr|\,\mathrm{d}x,$$

which together with (1) implies that μ is a distribution of order $\leqslant 1$, with

$$\operatorname{supp}\mu \subset \bigl[\inf W(x), \sup W(x)\bigr].$$

Consequently, $\mathrm{d}\rho * \mu$ is well defined in $\mathcal{D}'(\mathbf{R})$. Using (2), (5) and the definition of the convolution we get easily (4).

When $V = 0$, it was proved by Sjöstrand [4] that if $0 < E \in \operatorname{singsupp}_a(\mu)$, then the operator $P(h) = -\Delta + W(hx)$ has at least $C_\Omega h^{-n}$ resonances in any h-independent complex neighborhood Ω of E. Here $\operatorname{singsupp}_a(\mu)$ denotes the analytic singular support of the distribution μ.

Now let I be an open bounded interval. Assume that for all $\lambda \in I$ the following assumption holds.

(H2) For all $k_0 \in \mathbf{R}^n/\Gamma^*$ with $\lambda_i(k_0) = \lambda$, the eigenvalue $\lambda_i(k_0)$ is simple and $\mathrm{d}_k\lambda(k_0) \neq 0$.

The case $V \neq 0$ was recently studied by Dimassi and Zerzeri [1]. Under the assumption (H2) they obtained the same lower bound as in [4] near $E \in \mathrm{singsupp}_a(\omega) \cap I$. Surely, in this case ρ is more complicated and $\mathrm{singsupp}_a(\omega)$ will depend on both $\mathrm{singsupp}_a(\mu)$ and $\mathrm{singsupp}_a(\mathrm{d}\rho)$.

We recall that, when $V = 0$, $\rho(\lambda) = (2\pi)^{-n}\mathrm{vol}(B_{\mathbf{R}^n}(0,1))\max(\lambda, 0)^{n/2}$. This fact permitted to Sjöstrand to prove that $\mathrm{singsupp}_a(\mathrm{d}\rho * \mu) = \mathrm{singsupp}_a(\mu)$.

In this Note we will use the simple representation of ω given by Proposition 1 to get a lower bound near some singularities of $\rho(\lambda)$. More precisely we study resonances generated by analytic singularities of μ near the edge of bands or near some singularities of ρ due to the band crossings.

2. Lower bounds of the counting function near the edges of bands

The following result is a consequence of Morse lemma.

LEMMA 2. – *Let $e_0 \in \sigma(P_0)$. We assume that*:

(i) *If $\lambda_j(k) = e_0$, then $\lambda_j(k)$ is a simple eigenvalue of P_k.*

(ii) *There exist i_0 and k_0 such that $\lambda_{i_0}(k_0) = e_0$, $\nabla\lambda_{i_0}(k_0) = 0$, $\pm\partial^2\lambda_{i_0}(k_0) > 0$ and $\nabla\lambda_{i_0}(k) \neq 0$, $\forall k \in E^*$, $k \neq k_0$.*

(iii) *For all $k \in \lambda_i^{-1}\{e_0\}$ and all $i \neq i_0$, $\nabla\lambda_i(k) \neq 0$.*

Then there exists an open connected neighborhood J of e_0 such that

$$\rho(e) = f(e - e_0) + H\big(\pm(e - e_0)\big)g_\pm(\sqrt{e - e_0}), \quad \forall e \in J, \tag{6}$$

where f and $g_\pm$ are C^∞ and $g_\pm(0) = 0, \ldots, g_\pm^{(n-1)}(0) = 0$, $g_\pm^{(n)}(0) \neq 0$. Here, $+(-)$ corresponds to a local minimum (maximum respectively).

Using (4) and Lemma 2, we obtain:

THEOREM 3. – *Let e_0 and J be as above, and let $\lambda \in (e_0 + \mathrm{singsupp}_a(\mu))$. We assume that λ satisfies* (H2) *and that $(\lambda - \mathrm{supp}(\mu)) \subset J$. Then for all h-independent complex neighborhoods Ω of λ, there exist $h_0 = h(\Omega) > 0$ sufficiently small and $C = C(\Omega) > 0$ such that for $h \in]0, h_0[$,*

$$\#\big\{z \in \Omega;\, z \in \mathrm{Res}\big(P(h)\big)\big\} \geqslant C_\Omega h^{-n}.$$

Remark 4. – The assumption $(\lambda - \mathrm{supp}(\mu)) \subset J$, ensures that, in the study of $\mathrm{d}\rho * \mu$ near λ, one only needs the value of ρ in J given by (6). Hence, using (6) and Proposition 1, we show that $\lambda \in \mathrm{singsupp}_a(\omega)$. Therefore, Theorem 3 follows from the result of Dimassi and Zerzeri [1].

3. Lower bounds near singularities due to band crossings

In this subsection we study resonances near singularities of $\rho(\lambda)$ generated by a band crossings. We will only consider the two dimentional case. With similar assumptions, one can treat the case $n \geqslant 2$.

We assume that λ_j is a double eigenvalues $\lambda_{j-1}(k_0) < \lambda_j(k_0) = e_0 = \lambda_{j+1}(k_0) < \lambda_{j+2}(k_0)$ and that for all $k \neq k_0$ such that $\lambda_i(k) = e_0$, $\lambda_i(k)$ is simple and $\nabla\lambda_i(k) \neq 0$.

Since P_k is analytic in k, this implies that for $|k - k_0| \leqslant \delta$ (with δ small enough), the span $V(k)$, of the eigenvectors of P_k corresponding to eigenvalues in the set $\{e;\ |e - e_0| \leqslant \delta\}$ has a basis $\psi_j(x,k), \psi_{j+1}(x,k)$, which is orthonormal and real analytic in k. The restriction of P_k to $V(k)$ has the matrix

$$\begin{pmatrix} \alpha(k) & \overline{b(k)} \\ b(k) & \beta(k) \end{pmatrix},$$

which can be written

$$\begin{pmatrix} a(k)+c(k) & b_1(k)-\mathrm{i}b_2(k) \\ b_1(k)+\mathrm{i}b_2(k) & a(k)-c(k) \end{pmatrix},$$

where $a(k)=\alpha(k)+\beta(k)/2, c(k)=\alpha(k)-\beta(k)/2, b_1(k)$ and $b_2(k)$ are real valued. Next, the periodic potential is assumed to have the symmetry $V(x)=V(-x)$. This symmetry is typical of metals. This symmetry forces $b(k)$ to be real valued (i.e., $b_2(k)=0$). Consequently, near k_0 we have

$$\lambda_j(k)=a(k)-\sqrt{c^2(k)+b^2(k)}, \qquad \lambda_{j+1}(k)=a(k)+\sqrt{c^2(k)+b^2(k)}.$$

We assume that $\nabla b(k_0)$, $\nabla c(k_0)$ are independent. Since $n=2$, $(\nabla b(k_0),\nabla c(k_0))$ is a basis in $\mathbf{R}^2$. Set $\nabla a(k_0)=\alpha_1\nabla b(k_0)+\alpha_2\nabla c(k_0)$.

The following result was proved in [2].

LEMMA 5 ([2]). – *If $\alpha_1^2+\alpha_2^2<1$, then there exist an open connected neighborhood J of e_0 and* C^∞ *functions f and g such that*

$$\rho(e)=f(e)+\big(H(e-e_0)-H\big(-(e-e_0)\big)\big)g(e), \tag{7}$$

with $g''(e_0)\neq 0,\ \forall e\in J$.

THEOREM 6. – *Let J be an open interval in which* (7) *is valid. Let $\lambda\in(e_0+\mathrm{singsupp_a}(\mu))$ be satisfying* (H2). *We assume that $(\lambda-\mathrm{supp}(\mu))\subset J$. Then for all h-independent complex neighborhoods Ω of λ, there exist $h_0=h(\Omega)>0$ sufficiently small and $C=C(\Omega)>0$ such that for $h\in]0,h_0[$,*

$$\#\big\{z\in\Omega;\ z\in\mathrm{Res}\big(P(h)\big)\big\}\geqslant C_\Omega h^{-n}.$$

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