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Algebraic Geometry

Symplectic resolutions for coverings of nilpotent orbits

Résolutions symplectiques pour les revêtements des orbites nilpotentes

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Abstract

Let \mathcal{O} be a nilpotent orbit in a semisimple complex Lie algebra \mathfrak{g} . Denote by G the simply connected Lie group with Lie algebra \mathfrak{g} . For a G -homogeneous covering $M \rightarrow \mathcal{O}$, let X be the normalization of $\overline{\mathcal{O}}$ in the function field of M . In this Note, we study the existence of symplectic resolutions for such coverings X . **To cite this article:** B. Fu, C. R. Acad. Sci. Paris, Ser. I 336 (2003).

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Résumé

Soit \mathcal{O} une orbite nilpotente dans une algèbre de Lie semi-simple complexe \mathfrak{g} . Soit G le groupe de Lie simplement connexe d'algèbre de Lie \mathfrak{g} . Pour un revêtement G -homogène $M \rightarrow \mathcal{O}$, notons X la normalisation de $\overline{\mathcal{O}}$ dans le corps de fonctions de M . Dans cette Note, nous étudions les résolutions symplectiques pour de telles variétés X . **Pour citer cet article :** B. Fu, C. R. Acad. Sci. Paris, Ser. I 336 (2003).

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Version française abrégée

Soit G un groupe de Lie semi-simple complexe, connexe et simplement connexe d'algèbre de Lie \mathfrak{g} . Soit \mathcal{O} une orbite nilpotente dans \mathfrak{g} . Le groupe fondamental $\pi_1(\mathcal{O})$ de \mathcal{O} est fini et en général non nul (voir Corollaire 6.1.6 et Section 8.4 [2]). Soit $p : M \rightarrow \mathcal{O}$ un revêtement G -homogène. Notons X la normalisation de $\overline{\mathcal{O}}$ dans le corps de fonctions de M , qui est appelée un revêtement de $\overline{\mathcal{O}}$. Il a été démontré dans [1] que X contient M comme ouvert de Zariski et que le morphisme $p : M \rightarrow \mathcal{O}$ s'étend en un morphisme fini G -équivariant $\bar{p} : X \rightarrow \overline{\mathcal{O}}$. La forme de Kostant–Kirillov sur \mathcal{O} induit une forme symplectique ω sur M .

Définition 1. Une *résolution symplectique* de X est une résolution des singularités $\pi : Z \rightarrow X$ telle que π est projectif et que la 2-forme $\pi^*(\omega)$ définie a priori sur $\pi^{-1}(M)$ s'étend en une 2-forme symplectique sur Z tout entier.

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Dans cette Note, nous montrons le théorème suivant, qui généralise notre théorème principal dans [3].

Théorème 2. *Soit G un groupe de Lie semi-simple et simplement connexe d'algèbre de Lie \mathfrak{g} . Soit $\bar{p}: X \rightarrow \bar{\mathcal{O}}$ un revêtement de degré impair d . Supposons que $\pi: Z \rightarrow X$ soit une résolution symplectique. Alors il existe un sous-groupe parabolique P de G tel que Z soit isomorphe à $T^*(G/P)$ et que le morphisme $\bar{p} \circ \pi$ devienne $T^*(G/P) \simeq G \times^P \mathfrak{u} \rightarrow \mathfrak{g}$, $(g, u) \mapsto \text{Ad}(g)u$, où \mathfrak{u} est le nilradical de $\mathfrak{p} = \text{Lie}(P)$.*

Un corollaire amusant est le suivant

Corollaire 3. *Un revêtement de degré impair $d > 1$ d'une orbite nilpotente dans $\mathfrak{sl}(n, \mathbb{C})$ n'admet aucune résolution symplectique.*

1. Introduction

Let G be a simply connected semi-simple complex Lie group with Lie algebra \mathfrak{g} . Let \mathcal{O} be a nilpotent adjoint orbit in \mathfrak{g} . The fundamental group $\pi_1(\mathcal{O})$ of \mathcal{O} is finite and in general non-zero (see Corollary 6.1.6 and Section 8.4 of [2]). Let $p: M \rightarrow \mathcal{O}$ be a G -homogeneous covering of degree d . We denote by X the normalization of $\bar{\mathcal{O}}$ in the function field of M , which will be called a *covering* of the nilpotent orbit $\bar{\mathcal{O}}$. Then Proposition 1.2 of [1] says that X contains M as a Zariski open subset and the map p extends to a finite surjective G -equivariant morphism $\bar{p}: X \rightarrow \bar{\mathcal{O}}$. Furthermore G has finitely many orbits on X and $X - M$ is of codimension at least 2 in X . Recall that a regular 2-form on a smooth algebraic variety is *symplectic* if it is closed and non-degenerate at every point. The Kostant–Kirillov form on the orbit \mathcal{O} gives a symplectic form ω on M .

Definition 1.1. A *symplectic resolution* for X is a projective resolution of singularities $\pi: Z \rightarrow X$ such that the 2-form $\pi^*(\omega)$, defined a priori on $\pi^{-1}(M)$, extends to a symplectic form Ω on the whole of Z .

As shown in [3], a resolution is symplectic if and only if it is crepant. The purpose of this Note is to consider the existence of symplectic resolutions, i.e., for which coverings of a nilpotent adjoint orbit there exists a symplectic resolution. It turns out that this is a difficult problem. The following proposition gives some examples of symplectic resolutions.

Proposition 1.2. *Let P be a parabolic subgroup of G and M the unique open G -orbit in $T^*(G/P)$. Let \mathcal{O} be the orbit of a Richardson element in the nilradical \mathfrak{u} of the Lie algebra of P . Then the G -equivariant desingularization $T^*(G/P) \rightarrow X$ is a symplectic resolution for X .*

For the proof, see Proposition 7.4 of [1]. It should be pointed out that there exist some symplectic resolutions for coverings of nilpotent orbits which are not of the above form. An example is the double covering of the minimal nilpotent orbit in $\mathfrak{sp}(2n, \mathbb{C})$ which is $X = \mathbb{C}^{2n} \rightarrow \bar{\mathcal{O}}_{\min}$. However this is the only example where X is non-singular (see Theorem 4.6 [1]). The purpose of this note is to prove the converse of this proposition under some additional hypotheses.

Recall that \mathcal{O} and $\bar{\mathcal{O}}$ are both stable under the scaling action of \mathbb{C}^* on \mathfrak{g} . This induces the Euler action of \mathbb{C}^* on \mathcal{O} and $\bar{\mathcal{O}}$. Unfortunately this action does not lift to X in general. An example is the double covering of the minimal nilpotent orbit \mathcal{O}_{\min} in $\mathfrak{sp}(2n, \mathbb{C})$. However we have (see Lemma 1.3 and Proposition 1.4 [1])

Proposition 1.3. *For any covering X of $\bar{\mathcal{O}}$, there exists a \mathbb{C}^* -action on X which lifts the square of the Euler action of \mathbb{C}^* on $\bar{\mathcal{O}}$, i.e., for any $\lambda \in \mathbb{C}^*$ and $x \in X$, $\bar{p}(\lambda \cdot x) = \lambda^2 \bar{p}(x)$. Furthermore there exists a unique point $o \in X$ such that $\bar{p}(o) = 0$. This o is the unique G -fixed point in X and also the unique \mathbb{C}^* -fixed point in X .*

For this \mathbb{C}^* -action, we have $\lambda^* \omega = \lambda^2 \omega$ for any $\lambda \in \mathbb{C}^*$, which is different to our situation in [3], where the \mathbb{C}^* -action satisfies $\lambda^* \omega = \lambda \omega$. This makes the situation more complicated here. However, under the hypothesis that the degree of the covering is odd, we can prove the following

Theorem 1.4. *Let \mathfrak{g} be a semisimple complex Lie algebra and X a covering of a nilpotent adjoint orbit \mathcal{O} in \mathfrak{g} . Suppose that the degree d of the covering $\bar{p}: X \rightarrow \bar{\mathcal{O}}$ is odd. Then for any symplectic resolution $\pi: Z \rightarrow X$, Z is isomorphic to $T^*(G/P)$ for some parabolic subgroup P of G . Furthermore, under this isomorphism, the map $\bar{p} \circ \pi$ becomes $T^*(G/P) \simeq G \times^P \mathfrak{u} \rightarrow \mathfrak{g}$, $(g, u) \mapsto \text{Ad}(g)u$, where \mathfrak{u} is the nilradical of $\mathfrak{p} = \text{Lie}(P)$.*

This theorem generalizes our main theorem in [3], where we considered the case $d = 1$.

2. Outline of the proof

Lemma 2.1. *The Euler action of \mathbb{C}^* on $\bar{\mathcal{O}}$ lifts to X and for this action $\lambda^*\omega = \lambda\omega$.*

Proof. Let R be the regular functions ring of X . For $k \in \mathbb{Z}$, let $R[k] = \{\phi \in R \mid \lambda \cdot \phi = \lambda^k \phi, \lambda \in \mathbb{C}^*\}$. Proposition 1.4 of [1] implies that if the degree of the covering $X \rightarrow \bar{\mathcal{O}}$ is odd, then $R[k] = 0$ for k odd, i.e., $R = \sum_0^\infty R[2k]$. This gives that the Euler action of \mathbb{C}^* (not only its square!) on $\bar{\mathcal{O}}$ lifts to a \mathbb{C}^* -action on X . For this action, we have $\lambda^*\omega = \lambda\omega$. \square

Lemma 2.2. *The action of $G \times \mathbb{C}^*$ on X lifts to Z such that $\pi: Z \rightarrow X$ and $\bar{p} \circ \pi: Z \rightarrow \bar{\mathcal{O}}$ are $G \times \mathbb{C}^*$ -equivariant. For the \mathbb{C}^* -action on Z , we have $\lambda^*\Omega = \lambda\Omega$ for any $\lambda \in \mathbb{C}^*$.*

Note that the action of G (resp. \mathbb{C}^*) on M lifts to $\pi^{-1}(M) \subset Z$. Now the proof goes along the same line as Proposition 3.1 of [3]. Using the two lemmas, we can apply our analysis in [3] to complete the proof of Theorem 1.4. For reader's convenience, we give an outline of our method.

Let $Z^{\mathbb{C}^*}$ be the fixed points subvariety in Z . Since $\bar{p} \circ \pi$ is proper and the \mathbb{C}^* -action on $\bar{\mathcal{O}}$ extends to a \mathbb{C} -action, the valuative criterion of properness shows that there exists an attraction $q: Z \rightarrow Z^{\mathbb{C}^*}$. As show by Lemma 3.5 [3], the \mathbb{C}^* -action on the smooth connected component $Z_0 \subset Z^{\mathbb{C}^*}$ containing $q(\pi^{-1}(M))$ is definite, i.e., for $z \in Z_0$, $T_z^k Z = \{v \in T_z Z \mid \lambda_* v = \lambda^k v\}$ is zero if $k < 0$. Now the equation $\lambda^*\Omega = \lambda\Omega$ implies a duality between $T_z^k Z$ and $T_z^{1-k} Z$. Thus for $z \in Z_0$, we have $T_z Z_0 \simeq T_z^0 Z \simeq T_z^1 Z$ and $T_z^k Z = 0$ if $k \neq 0, 1$. This shows that Z_0 is Lagrangian in Z . By a classical result of Bialynicki-Birula, the attraction $q: q^{-1}(Z_0) \rightarrow Z_0$ is \mathbb{C}^* -equivariantly isomorphic to $T^*Z_0 \rightarrow Z_0$.

Now we study the G -action on Z_0 , which has an open-dense orbit, namely $q(\pi^{-1}(M))$. In fact this orbit is the whole of Z_0 . Note that $Z_0 \subset Z^{\mathbb{C}^*} \subset \pi^{-1}(o)$ is projective, so $Z_0 = G/P$ for some parabolic subgroup P of G . To complete the proof, we need to show that $Z_0 = Z^{\mathbb{C}^*}$. Note that $\pi^{-1}(o)$ is connected, so we need only to show that Z_0 is a connected component of $\pi^{-1}(o)$ or equivalently a connected component of $(\bar{p} \circ \pi)^{-1}(0)$, this is proved by using the explicit formula for the map $T^*(G/P) \rightarrow \bar{\mathcal{O}}$, $(g, u) \mapsto \text{Ad}(g)u$.

3. Some corollaries and examples

Corollary 3.1. *Let X be a covering of odd degree of a nilpotent orbit $\bar{\mathcal{O}}$. If \mathcal{O} is not a Richardson orbit, then X admits no symplectic resolution.*

This follows directly from the above theorem and Proposition 2. Consider the nilpotent orbits $2A_2, 2A_2 + A_1, A_5, E_6(a_1), E_6$ in the exceptional Lie algebra of type E_6 , here we use notations from Section 8.4 of [2]. All these nilpotent orbits have $\mathbb{Z}/3$ as fundamental group, but none of them is a Richardson orbit. The above corollary implies that the three-fold covering of any of the above orbits admits no symplectic resolution.

Corollary 3.2. *A covering X of odd degree $d > 1$ of a nilpotent orbit \mathcal{O} in $\mathfrak{sl}(n, \mathbb{C})$ admits no symplectic resolution.*

This follows from the above theorem and Theorem 3.3 [4], which says that for any polarization P of $\mathcal{O} \subset \mathfrak{sl}(n, \mathbb{C})$, the morphism $T^*(G/P) \rightarrow \bar{\mathcal{O}}$ is birational. Recall that every nilpotent orbit in $\mathfrak{sl}(n, \mathbb{C})$ admits a

symplectic resolution ([3]), however the above corollary shows that their odd higher degree coverings admit no symplectic resolution.

As an example, let \mathcal{O} be the principal nilpotent orbit in $\mathfrak{sl}(3, \mathbb{C})$, then $\pi_1(\mathcal{O}) = \mathbb{Z}/3$. Let X be the three-fold covering of $\overline{\mathcal{O}}$. Then by the above corollary, X does not admit any symplectic resolution. In fact, X is a so-called shared orbit in [1], more precisely X is nothing but the closure of the minimal nilpotent $\overline{\mathcal{O}}_{\min}$ in the exceptional Lie algebra \mathfrak{g}_2 , which is known to admit no symplectic resolution [3].

In the list of [1], shared orbits do not admit any symplectic resolution except in the following case: let \mathcal{O} be the nilpotent orbit in $\mathfrak{sp}(2n, \mathbb{C})$ corresponding to the partition $[2, 2, 1, \dots, 1]$. Then $\pi_1(\mathcal{O}) = \mathbb{Z}/2$. By results of [3], the nilpotent orbit $\overline{\mathcal{O}}$ does not admit any symplectic resolution. Let X be the double covering of $\overline{\mathcal{O}}$. Then X is isomorphic to $\overline{\mathcal{O}}_{\min}$ in $\mathfrak{sl}(2n, \mathbb{C})$, which admits a symplectic resolution.

As shown by the above examples, there exist some nilpotent orbits which admit some symplectic resolutions, but not their coverings, and there exist some nilpotent orbits which do not admit any symplectic resolution, while some of their coverings do admit some symplectic resolutions. This indicates that the problem to determine which covering admits a symplectic resolution might be difficult and interesting.

4. A conjecture

From now on, we suppose that \mathfrak{g} is a simple complex Lie algebra. Let \mathcal{O} be a nilpotent adjoint orbit in \mathfrak{g} and X a covering of $\overline{\mathcal{O}}$. Recall that $R[k] = \{\phi \in R \mid \lambda \cdot \phi = \lambda^k \phi, \lambda \in \mathbb{C}^*\}$. Let $\mathfrak{g}' = R[2]$, which is also a simple Lie algebra (see Theorem 4.2 of [1]). Let G' be the simply connected Lie group with Lie algebra \mathfrak{g}' , which is the maximal connected Lie group of holomorphic symplectic automorphisms of X (see Corollary 7.3 [1]). By Theorem 3.1 [1], X contains a Zariski open subset M' such that $M \subset M'$, and there exists a finite G' -covering $M' \rightarrow \mathcal{O}'$ to a nilpotent orbit \mathcal{O}' in \mathfrak{g}' . We propose the following conjecture on symplectic resolutions for X :

Conjecture 1. *Let $\bar{p}: X \rightarrow \overline{\mathcal{O}}$ be a covering of a nilpotent orbit in a simple Lie algebra \mathfrak{g} such that $R[1] = 0$. Suppose that $\pi: Z \rightarrow X$ is a symplectic resolution. Then there exists some parabolic subgroup P' of G' such that Z is isomorphic to $T^*(G'/P')$ and the map π becomes the collapsing of the zero section $T^*(G'/P') \rightarrow X$.*

Remark 1. It is proved in [1] (Theorem 4.6) that $R[1] \neq 0$ if and only if X is the double covering of $\overline{\mathcal{O}}_{\min} \subset \mathfrak{sp}(2n, \mathbb{C})$. In this case, $X = \mathbb{C}^{2n}$.

Remark 2. By Theorem 7.5 [1], if $T^*(G/P) \rightarrow X$ is a symplectic resolution for X , then there exists some parabolic subgroup P' of G' such that $G/P = G'/P'$.

Remark 3. If the covering $\bar{p}: X \rightarrow \overline{\mathcal{O}}$ is of odd degree, then by our Theorem 1.4, the conjecture is true.

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