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Partial Differential Equations

Large time behaviour of solutions of the Swift–Hohenberg equation

Comportement des solutions de l'équation de Swift–Hohenberg en grand temps

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Abstract

We study the limiting profiles v of solutions of the Swift–Hohenberg equation on a one-dimensional domain $(0, L)$ for different values of L. We identify those values of L for which $v = 0$, and discuss the size and the shape of v when it is nontrivial and a global minimiser of an associated energy functional. ***To cite this article: L.A. Peletier, V. Rottschäfer, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Nous étudions les limites des profiles v des solutions de l'équation Swift–Hohenberg dans une domaine de dimension un $(0, L)$, pour différents choix de L. Nous identifions les valeurs de L pour lesquelles $v = 0$ et nous derivons des estimations pour la taille et la forme quand v minimise une fonctionnelle associée. ***Pour citer cet article : L.A. Peletier, V. Rottschäfer, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

Nous considérons le problème de Cauchy–Dirichlet pour l'équation de Swift–Hohenberg :

$$\begin{cases} \frac{\partial u}{\partial t} = \alpha u - \left(1 + \frac{\partial^2}{\partial x^2}\right)^2 u - u^3, & 0 < x < L,\ t > 0, \\ u = 0 \quad \text{et} \quad \frac{\partial^2 u}{\partial x^2} = 0, & x = 0, L,\ t > 0, \\ u(x, 0) = u_0(x), & 0 < x < L, \end{cases} \tag{I}$$

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où $\alpha \in (0,1)$ et $L > 0$ sont des constantes, et $u_0(\frac{1}{2}L - x) = u_0(\frac{1}{2}L + x)$. On définit des longueurs critiques :

$$L_1(\alpha) = \frac{\pi}{\sqrt{1+\sqrt{\alpha}}} \quad \text{et} \quad L_2(\alpha) = \frac{\pi}{\sqrt{1-\sqrt{\alpha}}}.$$

Evidemment, $L_{1,2}(\alpha) \to \pi$ quand $\alpha \to 0$, et

$$(2n-1)L_2 < (2n+1)L_1 \quad \text{si } \alpha < \alpha_n \stackrel{\text{def}}{=} 4\left\{2n + (2n)^{-1}\right\}^{-2}, \ n = 1, 2, \ldots.$$

Théorème 1. *Soit $u(t)$ la solution du problème* (I). *On suppose $\alpha < \alpha_n$ pour un certain $n \geqslant 1$. Alors on a :*

$$u(t) \to 0 \quad \textit{quand } t \to \infty,$$

si $L \in (0, L_1] \cup [L_2, 3L_1] \cup \cdots \cup [(2n-1)L_2, (2n+1)L_1]$.

Si $u(t) \to v$ quand $t \to \infty$ il est bien connu que v est un point critique de la fonctionnelle :

$$J(u, L) = \frac{1}{L}\int_0^L \left\{\frac{1}{2}(u'')^2 - (u')^2 + F_\alpha(u)\right\} \mathrm{d}x,$$

si les points critiques de J sont isolés.

Théorème 2. *On suppose que*

$$J(v, L) = \min\left\{J(u, L)\colon\ u \in X\right\},$$

où $X = \{u \in H^2 \cap H_0^1(0, L)\colon\ u(\frac{1}{2}L - x) = u(\frac{1}{2}L + x)\}$. Alors on a :

$$\left(1 - \frac{1}{\sqrt{3}}\right)\left|P\left(\frac{\pi}{L}\right)\right| \leqslant \frac{1}{L}\int_0^L v^2(x)\,\mathrm{d}x \leqslant \left(1 + \frac{1}{\sqrt{3}}\right)\left|P\left(\frac{\pi}{L}\right)\right| \quad \textit{si } L \in \omega_1,$$

où la fonction $P(\xi)$ est le symbole de l'opérateur linéaire associé à l'équation de Swift–Hohenberg linéarisée : $P(\xi) = (\xi^2 - 1)^2 - \alpha$, *et $\omega_1 = (L_1, \ell)$ où $\ell = \sup\{L > L_1\colon\ P(\pi/L) < P(k\pi/L),\ k = 2, 3, \ldots\}$.*

In studies of pattern formation, the *Swift–Hohenberg equation*

$$\frac{\partial u}{\partial t} = \alpha u - \left(1 + \frac{\partial^2}{\partial x^2}\right)^2 u - u^3, \quad \alpha \in \mathbf{R}, \tag{SH}$$

plays a central role. Proposed in 1987 by Swift and Hohenberg [11] in connection with Rayleigh–Bénard convection, it has since featured in a variety of problems, such as Taylor–Couette flow [5,7], and in the study of lasers [6]. We also refer to the surveys given in [3] and [2] and the recent review [1].

The Swift–Hohenberg equation is interesting from the point of view of pattern formation, because it has many qualitatively different stable equilibrium solutions (cf. [12] and the survey given in [9]). This begs the question, which of these equilibrium solutions will be selected as time tends to infinity and it is one of the questions we address in this Note.

To be specific we study the Cauchy–Dirichlet problem for the Swift–Hohenberg equation (SH) on the interval $(0, L)$, where the length L will be an important parameter. Thus, writing Eq. (SH) in more conventional form, we consider the problem:

$$\begin{cases} u_t = -u_{xxxx} - 2u_{xx} - f_\alpha(u) & \text{for } 0 < x < L,\ t > 0, \\ u = 0 \quad \text{and} \quad u_{xx} = 0 & \text{for } x = 0, L,\ t > 0, \\ u(x, 0) = u_0(x) & \text{for } 0 < x < L, \end{cases} \tag{1}$$

where $f_\alpha(s) = s^3 + (1-\alpha)s$, and u_0 is a smooth symmetric function, i.e., $u_0(\frac{1}{2}L - x) = u_0(\frac{1}{2}L + x)$. Note that this assumption implies that the solution remains symmetric for all t. This problem is a *Gradient System* with corresponding functional

$$J(u,L) = \frac{1}{L}\int_0^L \left\{\frac{1}{2}(u'')^2 - (u')^2 + F_\alpha(u)\right\} \mathrm{d}x,$$

where

$$F_\alpha(u) = \int_0^u f_\alpha(s)\,\mathrm{d}s = \frac{1-\alpha}{2}u^2 + \frac{1}{4}u^4.$$

Therefore, if the stationary solutions of problem (1) are isolated, then $u(x,t)$ tends to one of these solutions as $t \to \infty$ (cf. [4]), in other words for every $x \in (0, L)$:

$$u(x,t) \to v(x) \quad \text{as } t \to \infty,$$

where $v(x)$ is a solution of the two-point boundary value problem:

$$\begin{cases} v^{(\mathrm{iv})} + 2v'' + v^3 + (1-\alpha)v = 0 & \text{for } 0 < x < L, \\ v = 0 \quad \text{and} \quad v'' = 0 & \text{at } x = 0, L. \end{cases} \tag{2}$$

It is known that if $\alpha \leqslant 0$, then problem (2) has the trivial solution only (cf. Chapter 9 of [9]), and hence, for every $x \in (0, L)$, $u(x,t) \to 0$ as $t \to \infty$. For $\alpha > 0$ the situation is much more complicated and, depending on the value of α and of L, there may be very many profiles to choose from.

In Figs. 1 and 2 we show results of numerical computations in which we have taken $\alpha = 0.3$ and the initial function as

$$u_0(x) = A\sin\left(\frac{\pi x}{L}\right), \quad A = \frac{1}{10}. \tag{3}$$

In Fig. 1 we see how, depending on the value of L, the solution evolves into different periodic patterns as $t \to \infty$: for $L = 3$ the final profile v is qualitatively similar to that of the initial profile, and half a period of a sine-like profile fills the entire domain $(0, L)$. Then, for $L = 6$ something surprising happens: the final profile is the trivial solution $v = 0$, as it is for L small enough. When L is increased to $L = 9$ the final profile is nontrivial again, however, now it is a periodic profile that packs *three* half periods into $(0, L)$.

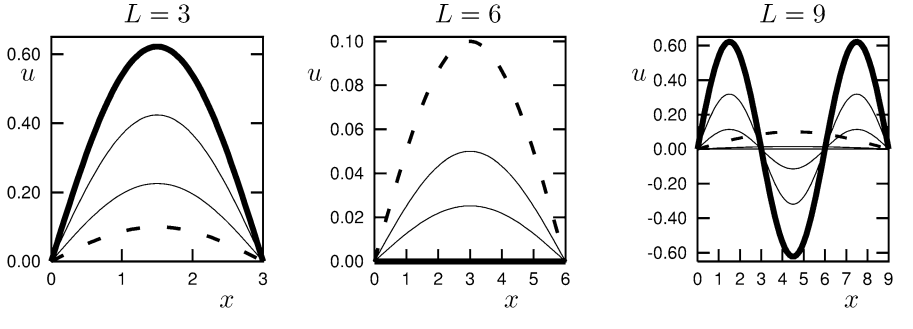

Fig. 1. Profiles for $\alpha = 0.3$ and different choices of $L = 3, 6$ and 9. The dotted curve is the initial profile (3) and the thick curve the final profile. The other curves represent profiles at intermediate times.

Fig. 1. Profiles pour $\alpha = 0{,}3$ et $L = 3, 6$ et 9 : la courbe pointillé est le profil initial (3), et la courbe « grasse » le profil final. Les autres courbes sont des profiles aux temps intermédiaires.

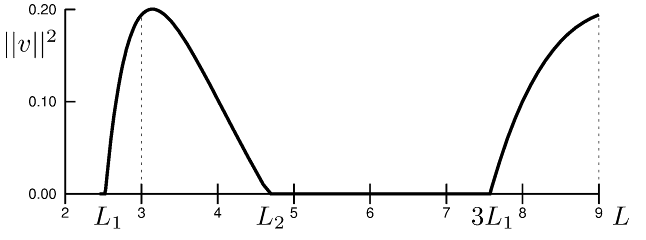

Fig. 2. Graph of the L^2-norm of the final profile for $\alpha = 0.3$ and $L \in [2, 9]$.

Fig. 2. Graphe de la norme L^2 du profile final pour $\alpha = 0{,}3$ et $L \in [2, 9]$.

In Fig. 2 we show how the L^2-norm of the final profile v, scaled to take into account the length of the interval,

$$\|v\|^2 = \frac{1}{L}\int_0^L v^2(x)\,\mathrm{d}x,$$

varies with respect to L. We see that there exist critical values L_1 and L_2 of L, such that for $L \leqslant L_1$ and $L_2 \leqslant L \leqslant 3L_1$ the final profile v is identically zero, whilst for $L \in (L_1, L_2)$ and for larger values of L it is not.

We shall prove that at the critical lengths, $L_1(\alpha)$ and $L_2(\alpha)$, which are given by

$$L_1(\alpha) = \frac{\pi}{\sqrt{1+\sqrt{\alpha}}} \quad \text{and} \quad L_2(\alpha) = \frac{\pi}{\sqrt{1-\sqrt{\alpha}}},$$

branches of nontrivial solutions of problem (2) bifurcate from the trivial solution.

Theorem 1. *At each of the points* $(L_1, 0)$ *and* $(L_2, 0)$ *in the* (L, v)*-plane, a branch of nontrivial solutions of problem* (2) *bifurcates from the trivial solution. The local behaviour of these branches is given by*:

$$\|v\|^2 \sim \frac{3}{8\pi}\sqrt{\alpha}\,(1+\sqrt{\alpha})^{3/2}(L-L_1) \quad \textit{as } L \searrow L_1, \qquad \|v\|^2 \sim \frac{3}{8\pi}\sqrt{\alpha}\,(1-\sqrt{\alpha})^{3/2}(L_2-L) \quad \textit{as } L \nearrow L_2.$$

In addition, we prove that if $L_2(\alpha) < 3L_1(\alpha)$, which is the case when $0 < \alpha < \frac{16}{25}$, then

$$u(t) \to 0 \quad \text{as } t \to \infty \text{ when } L \in [L_2, 3L_1].$$

This confirms what was observed in Figs. 1 and 2.

More generally we have:

$$(2n-1)L_2 < (2n+1)L_1 \quad \text{if } \alpha < \alpha_n \stackrel{\text{def}}{=} 4\bigl\{2n + (2n)^{-1}\bigr\}^{-2},\ n = 1, 2, \ldots,$$

and we prove the following result for values of L in the set

$$\Sigma_n \stackrel{\text{def}}{=} L \in (0, L_1] \cup [L_2, 3L_1] \cup \cdots \cup \bigl[(2n-1)L_2, (2n+1)L_1\bigr].$$

Theorem 2. *Let* $u(t)$ *be the solution of problem* (1), *in which* $\alpha < \alpha_n$ *for some* $n \geqslant 1$. *Then, whenever* $L \in \Sigma_n$,

$$u(t) \to 0 \quad \textit{as } t \to \infty.$$

These results can be understood in terms of the linear eigenvalue problem obtained from problem (1) by substituting $u(x, t) = \mathrm{e}^{-\lambda t}\varphi(x)$ and omitting the nonlinear term:

$$\begin{cases} \varphi^{(\mathrm{iv})} + 2\varphi'' + (1-\alpha)\varphi = \lambda\varphi & \text{for } x \in (0, L), \\ \varphi = 0 \quad \text{and} \quad \varphi'' = 0 & \text{at } x = 0,\ L. \end{cases} \tag{4}$$

An elementary computation shows that the eigenfunctions and the eigenvalues are given by:

$$\varphi_n(x) = \sqrt{2}\sin\left(\frac{n\pi x}{L}\right) \quad \text{and} \quad \lambda_n = \lambda_n(L) = P\left(\frac{n\pi}{L}\right),$$

where $P(\xi)$ is the symbol of the equation in (4): $P(\xi) = (\xi^2 - 1)^2 - \alpha$. The eigenfunctions are orthonormal, and have been normalised so that $\|\varphi_n\| = 1$. In Fig. 3 we give graphs of $P(\xi)$ and $P(n\pi/L)$. The critical lengths L_1 and L_2 correspond to the zeros of λ_1, i.e., of $P(\pi/L)$. Thus, $L_1 = \pi/\xi_+$ and $L_2 = \pi/\xi_-$, where $\xi_\pm$ are the zeros of P. Plainly, the zeros of λ_n are nL_1 and nL_2.

The linear stability of the trivial solution depends on the smallest eigenvalue:

$$\mu(L) = \inf\{\lambda_n(L)\colon\ n \geqslant 1\}.$$

One finds that if $\alpha \leqslant \alpha_n$, then $\mu(L) \geqslant 0$ for $L \in \Sigma_n$, and $u = 0$ is linearly stable. Energy estimates show that this implies that $u = 0$ is a global attractor.

In the case that $\mu < 0$, one can distinguish intervals ω_n in which $\mu = \lambda_n$ and $\lambda_n < 0$, $n = 1, 2, \ldots$ (see Fig. 3). Note that $\omega_1 = (L_1, L_2)$ if $\alpha \in (0, \frac{16}{25})$.

Problem (1) being a gradient system, the solution $u(t)$ of problem (1) often converges to a global minimiser of the problem

$$\min\{J(u, L)\colon\ u \in X\}, \tag{5}$$

where $X = H^2 \cap H_0^1(0, L)$. The existence of such a minimiser is well known for any $L > 0$ and $\alpha \in \mathbf{R}$ [10]. Here we make the additional assumption that X contains the *symmetric* functions only, i.e., we assume that functions $w \in X$ are endowed with the property $w(\frac{1}{2}L - x) = w(\frac{1}{2}L + x)$.

We prove the following bounds:

Theorem 3. *Suppose that u is a global minimiser of problem* (5) *and that $\alpha \in (0, 1)$. Then,*

$$J(u, L) \leqslant -\frac{1}{6}\left|P\left(\frac{\pi}{L}\right)\right|^2 \quad \text{for } L \in (L_1, L_2), \qquad J(u, L) \geqslant -\frac{1}{4}\left|P\left(\frac{\pi}{L}\right)\right|^2 \quad \text{for } L \in \omega_1,$$

and

$$\left(1 - \frac{1}{\sqrt{3}}\right)\left|P\left(\frac{\pi}{L}\right)\right| \leqslant \|u\|^2 \leqslant \left(1 + \frac{1}{\sqrt{3}}\right)\left|P\left(\frac{\pi}{L}\right)\right| \quad \text{for } L \in \omega_1.$$

In Fig. 4 we show a graph of $J(v, L)$ versus L for the final profile v, obtained in numerical simulations with $\alpha = 0.3$, and u_0 given by (3). We conjecture that in this case v is the global minimiser of (5).

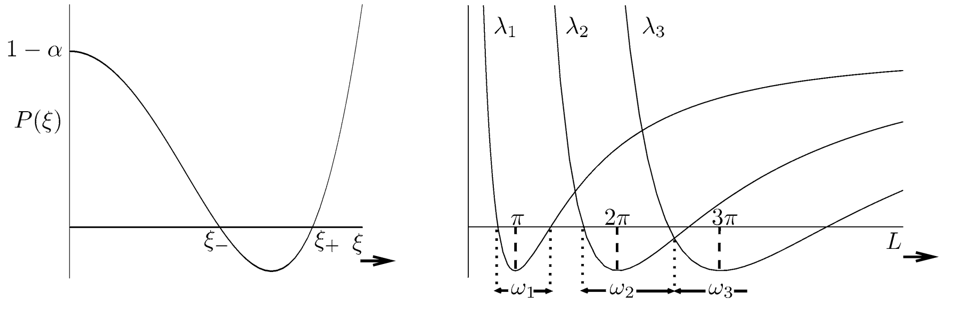

Fig. 3. Graphs of the polynomial $P(\xi)$ and of the eigenvalues $\lambda_n = P(n\pi/L)$ as a function of L for $n = 1, 2$ and 3.

Fig. 3. Graphes du polynomial $P(\xi)$ et des valeurs propres $\lambda_n = P(n\pi/L)$ comme fonction de L pour $n = 1, 2$ et 3.

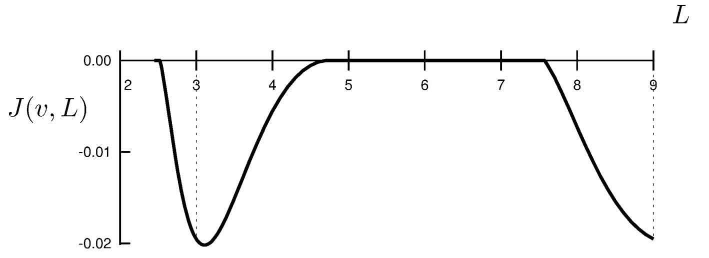

Fig. 4. Graph of the action $J(v, L)$ of the final profile for $\alpha = 0.3$ and $L \in [2, 9]$.

Fig. 4. Graphe de l'action $J(v, L)$ du profile final pour $\alpha = 0{,}3$ et $L \in [2, 9]$.

We conclude with an observation about the *shape* of the minimiser when $L \in \omega_1$. Numerical results reveal a remarkable likeness of quite large minimisers to an appropriate multiple of the eigenfunction φ_1. The following result shows that this is no mere coincidence. We write the minimiser u as $u = a_1\varphi_1 + w,\ a_1 = (u, \varphi_1)$. Then w lies in the orthogonal complement of the space $\mathcal{S}_1 = \{t\varphi_1\colon\ t \in \mathbf{R}\}$, and $\|w\|/a_1$ is the tangent of the angle θ between u and φ_1 and so is a measure for the distance between u and $\mathcal{S}_1$. For this quantity we find the following bound:

Theorem 4. *Let u be a global minimiser of problem* (5) *in which $\alpha \in (0, 1)$, and let θ be the angle between u and φ_1. Then*

$$\tan^2\theta = \frac{\|w\|^2}{a_1^2} \leqslant \frac{1}{4}\sqrt{3}(\sqrt{3}+1)\frac{|P(\pi/L)|}{P(3\pi/L)} \quad \textit{when } L \in \omega_1.$$

Example. We choose $\alpha = 0.3$ and $L = \pi$, so that the global minimiser u of $J(u, L)$ is quite large. The above bound then shows that $\theta \leqslant 0.0745\ldots$. Hence u is still very close to $\mathcal{S}_1$, implying that it is close to an appropriate multiple of φ_1.

Detailed proofs of Theorems 1, 2, 3 and 4 will appear in [8].

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