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Numerical Analysis/Calculus of Variations

Numerical methods for the solution of a system of Eikonal equations with Dirichlet boundary conditions

Méthode numériques pour la résolution d'un système d'équations eiconales avec conditions aux limites de Dirichlet

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Abstract

In this Note, we discuss the numerical solution of a system of Eikonal equations with Dirichlet boundary conditions. Since the problem under consideration has infinitely many solutions, we look for those solutions which are nonnegative and of maximal (or nearly maximal) L^1-norm. The computational methodology combines penalty, biharmonic regularization, operator splitting, and finite element approximations. Its practical implementation requires essentially the solution of cubic equations in one variable and of discrete linear elliptic problems of the Poisson and Helmholtz type. As expected, when the spatial domain is a square whose sides are parallel to the coordinate axes, and when the Dirichlet data vanishes at the boundary, the computed solutions show a fractal behavior near the boundary, and particularly, close to the corners. ***To cite this article: B. Dacorogna et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Dans cette Note, on étudie la résolution numérique d'un système d'équations eiconales avec conditions aux limites du type Dirichlet. Dans la mesure, où le problème considéré a une infinité de solutions on recherche celles qui sont non-négatives et de norme L^1 maximale (on presque maximale). La méthodologie numérique combine pénalité, régularisation biharmonique, décomposition d'opérateurs, et approximations par éléments finis. Son implémentation demande essentiellement la résolution d'équations à une variable du troisième degré et de problèmes linéaires elliptiques discrets pour le Laplacien et l'opérateur d'Helmholtz. Comme prévu, quand le domaine spatial est un carré de côtés parallèles aux axes de coordonnées les solutions calculées montrent un comportement fractal au voisinage de la frontière et plus particulièrement des coins. ***Pour citer cet article : B. Dacorogna et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

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L'analyse de modèles non linéaires en Mécanique et Science des Matériaux a conduit le premier auteur de cette Note à considérer le système de type *eiconal* ci-dessous,

$$u \in H_0^1(\Omega), \quad \left|\frac{\partial u}{\partial x_i}\right| = 1 \quad \text{p.p. dans } \Omega,\ i = 1, \ldots, d, \tag{1}$$

où Ω est un domaine borné de $\mathbb{R}^d$ $(d \geqslant 1)$ de fontière Γ. Dans la mesure où (1) a une infinité de solutions on va se restreindre à celles qui *maximisent* (où « presque » maximisent) la fonctionelle $v \to \int_\Omega v\,\mathrm{d}x$ $(\mathrm{d}x = \mathrm{d}x_1 \cdots \mathrm{d}x_d)$. Ceci conduit au problème du *Calcul des Variations*

$$u \in E; \quad \int_\Omega u\,\mathrm{d}x \geqslant \int_\Omega v\,\mathrm{d}x, \quad \forall v \in E, \tag{2}$$

où $E = \{v \mid v \in H_0^1(\Omega),\ |\partial v/\partial x_i| = 1 \text{ p.p.}, \forall i = 1, \ldots, d\}$. Problème (2) n'a pas de solution en général ; si cependant une telle solution existe elle vérifie $u \geqslant 0$. Comme par ailleurs $|\nabla v|^2 = d$, $\forall v \in E$, (2) est equivalent à

$$u \in E^+, \quad J(u) \leqslant J(v), \quad \forall v \in E^+, \tag{3}$$

où $E^+ = \{v \mid v \in E,\ v \geqslant 0 \text{ on } \overline{\Omega}\}$ et $J(v) = (1/2)\int_\Omega |\nabla v|^2\,\mathrm{d}x - C\int_\Omega v\,\mathrm{d}x$, C étant une constante positive arbitraire. Pour résoudre (3) numériquement on l'approche, avec $\varepsilon = \{\varepsilon_1, \varepsilon_2\}$, $\varepsilon_i > 0$, $\forall i = 1, 2$, par

$$u_\varepsilon \in K^+, \quad J^\varepsilon(u_\varepsilon) \leqslant J^\varepsilon(v), \quad \forall v \in H^2(\Omega) \cap K^+, \tag{4}$$

où

$$K^+ = \left\{v \mid v \in H_0^1(\Omega),\ v \geqslant 0 \text{ p.p. dans } \Omega\right\}$$

et

$$J^\varepsilon(v) = \frac{\varepsilon_1}{2}\int_\Omega |\Delta v|^2\,\mathrm{d}x + J(v) + \frac{\varepsilon_2^{-1}}{4}\sum_{i=1}^{d}\int_\Omega \left(\left|\frac{\partial v}{\partial x_i}\right|^2 - 1\right)^2 \mathrm{d}x.$$

Par des arguments de compacité-convexité on pent montrer que le problème – de type obstacle – (4) admet au moins une solution. L'approximation de (4) par éléments finis et la résolution du problème discret correspondant par des méthodes de *décomposition d'opérateurs* du type *Marchuk–Yanenko*, qui rammènent la résolution approchée de (4) à celle d'une suite de problèmes elliptiques linéaires à coefficients constants, font l'objet des Paragraphes 3, 4, et 5. Les résultats d'essais numériques sont donnés au Paragraphe 6 ; ils montrent que lorsque le problème (2) n'a pas de solution, les solutions du problème regularisé ont un comportement fractal au voisinage de Γ.

1. Introduction

Motivated by the analysis of nonlinear models from *Mechanics* and *Material Science*, the first author of this note has been lead to investigate (see [1] for further details) the properties of the solution of the following nonlinear boundary value problem:

$$\text{Find } u \in H_0^1(\Omega) \text{ such that } \left|\frac{\partial u}{\partial x_i}\right| = 1 \text{ a.e. in } \Omega,\ \forall i = 1, \ldots, d, \tag{1}$$

where, in (1), Ω is a bounded domain of $\mathbb{R}^d$ $(d \geqslant 1)$ of boundary Γ; problem (1) can be viewed as a boundary value problem for a *system of Eikonal equations*. The main goal of this Note is to describe a method for the numerical solution of (1), based on *operator-splitting* and *linear elliptic solvers* (after an appropriate *finite element* discretization). The method will be applied to the solution of two-dimensional test problems where $\Omega = (0, 1) \times (0, 1)$.

2. Formulation of variational problems related to (1)

Since problem (1) has infinitely many solutions, we are going to restrict our attention to those solutions which *maximize* (or "nearly" maximize) functional $v \to \int_\Omega v\,\mathrm{d}x$ (with $\mathrm{d}x = \mathrm{d}x_1 \cdots \mathrm{d}x_d$). This leads us to consider the following problem of the *Calculus of Variations*:

$$u \in E; \quad \int_\Omega u\,\mathrm{d}x \geqslant \int_\Omega v\,\mathrm{d}x, \quad \forall v \in E, \tag{2}$$

with $E = \{v \mid v \in H_0^1(\Omega),\ |\partial v/\partial x_i| = 1 \text{ a.e.}, \forall i = 1, \ldots, d\}$. If $d = 2$ and $\Omega = \{x \mid x = \{x_1, x_2\},\ |x_1 \pm x_2| < 1\}$, the *unique* solution of problem (2) is clearly given by $u(x) = 1 - |x_1| - |x_2|,\ \forall x \in \Omega$; on the other hand, problem (2) has no solution in general. We can easily show that if u is a solution of (2), then $u(x) \geqslant 0$ on $\overline{\Omega}$, moreover, since $|\nabla v|^2 = d$ a.e. on Ω if $v \in E$, problem (2) is equivalent to

$$u \in E^+; \quad J(u) \leqslant J(v), \quad \forall v \in E^+, \tag{3}$$

where $E^+ = \{v \mid v \in E,\ v \geqslant 0 \text{ on } \overline{\Omega}\}$, and $J(v) = (1/2)\int_\Omega |\nabla v|^2\,\mathrm{d}x - C\int_\Omega v\,\mathrm{d}x$, C being an (arbitrary) positive constant. There are several (if not many) ways to solve *numerically* problem (2), (3); motivated by some earlier work on the numerical solution of a *Ginzburg–Landau* equation [4,8] we are going to treat the constraints $|\partial v/\partial x_i| = 1$, $\forall i = 1, \ldots, d$, by a (exterior) *penalty* method. Moreover, in order to "control" the mesh related oscillations, we are going to bound $\|\Delta v\|_{L^2(\Omega)}$. This leads us to approximate $J(\cdot)$ by $J^\varepsilon(\cdot)$ defined as follows:

$$J^\varepsilon(v) = \frac{\varepsilon_1}{2}\int_\Omega |\Delta v|^2\,\mathrm{d}x + J(v) + \frac{\varepsilon_2^{-1}}{4}\sum_{i=1}^d \int_\Omega \left(\left|\frac{\partial v}{\partial x_i}\right|^2 - 1\right)^2 \mathrm{d}x, \tag{4}$$

where $\varepsilon = \{\varepsilon_1, \varepsilon_2\}$ with $\varepsilon_1, \varepsilon_2 > 0$ and "small". We then approximate problem (3) by:

$$u_\varepsilon \in K^+; \quad J^\varepsilon(u_\varepsilon) \leqslant J^\varepsilon(v), \quad \forall v \in H^2(\Omega) \cap K^+, \tag{5}$$

with $K^+ = \{v \mid v \in H_0^1(\Omega),\ v \geqslant 0 \text{ a.e. on } \Omega\}$. Proving that problem (5) has at least a solution is an (almost) elementary exercise. The *numerical solution* of (the *obstacle*) problem (5) is the main objective of this Note.

Remark 2.1. *Augmented Lagrangian methods* (closely related to those discussed in, e.g., [6]) can be applied to the solution of (5). Preliminary results look promising and will be reported elsewhere.

Remark 2.2. Suppose that we look for a solution of (1) as close as possible (in $H_0^1(\Omega)$) of $\varphi_d \in H_0^1(\Omega)$; in that case, we should take, in (3), (4), $J(\cdot)$ defined by $J(v) = (1/2)\int_\Omega |\nabla v|^2\,\mathrm{d}x - \int_\Omega \nabla\varphi_d \cdot \nabla v\,\mathrm{d}x$.

3. An equivalent variational formulation of problem (5) and an associated initial value problem

Let us denote $(L^2(\Omega))^d$ by Λ and ∇u_ε by $\mathbf{p}^\varepsilon$; problem (5) is clearly equivalent to

$$\mathbf{p}^\varepsilon \in \Lambda; \quad j^\varepsilon(\mathbf{p}^\varepsilon) \leqslant j^\varepsilon(\mathbf{q}), \quad \forall \mathbf{q} \in \Lambda, \tag{6}$$

with $\mathbf{q} = \{q_i\}_{i=1}^d$, and, $\forall \mathbf{q} \in \Lambda$,

$$j^\varepsilon(\mathbf{q}) = \frac{1}{2}\int_\Omega |\mathbf{q}|^2\,\mathrm{d}x - C\int_\Omega \nabla\varphi_1 \cdot \mathbf{q}\,\mathrm{d}x + \frac{\varepsilon_2^{-1}}{4}\sum_{i=1}^d \int_\Omega \left(|q_i|^2 - 1\right)^2 \mathrm{d}x + I_+(\mathbf{q}); \tag{7}$$

in (7), φ_1 is the unique solution in $H_0^1(\Omega)$ of $-\Delta\varphi_1 = 1$ in Ω, $\varphi_1 = 0$ on Γ, and $I_+(\cdot)$ is the functional defined as follows: $I_+(\mathbf{q}) = (\varepsilon_1/2)\int_\Omega |\nabla \cdot \mathbf{q}|^2\,\mathrm{d}x$ if $\mathbf{q} \in \nabla(H^2(\Omega) \cap K^+)$, $I_+(\mathbf{q}) = +\infty$ elsewhere. It is easy to show that

$I_+(\cdot)$ is *convex, proper* and *lower semi-continuous* over space Λ. In *variational form*, the (formal) *Euler–Lagrange equation* associated to problem (6) can be expressed as follows:

$$\int_\Omega \mathbf{p}^\varepsilon \cdot \mathbf{q}\,\mathrm{d}x + \varepsilon_2^{-1}\sum_{i=1}^d \int_\Omega \big(|p_i^\varepsilon|^2 - 1\big) p_i^\varepsilon q_i\,\mathrm{d}x + \big\langle \partial I_+(\mathbf{p}^\varepsilon),\ \mathbf{q}\big\rangle = C\int_\Omega \nabla\varphi_1\cdot\mathbf{q}\,\mathrm{d}x, \quad \forall \mathbf{q}\in\Lambda;\ \mathbf{p}^\varepsilon\in\Lambda, \tag{8}$$

with $\partial I_+(\cdot)$ the *subgradient* of $I_+(\cdot)$. We associate to Eq. (8) the following *initial value problem*

$$\begin{cases} \displaystyle\int_\Omega \left(\frac{\partial \mathbf{p}^\varepsilon}{\partial t}\right)\cdot\mathbf{q}\,\mathrm{d}x + \int_\Omega \mathbf{p}^\varepsilon\cdot\mathbf{q}\,\mathrm{d}x + \varepsilon_2^{-1}\sum_{i=1}^d\int_\Omega \big(|p_i^\varepsilon|^2-1\big)p_i^\varepsilon q_i\,\mathrm{d}x + \big\langle\partial I_+(\mathbf{p}^\varepsilon),\mathbf{q}\big\rangle \\ \quad = C\displaystyle\int_\Omega \nabla\varphi_1\cdot\mathbf{q}\,\mathrm{d}x, \quad \forall\mathbf{q}\in\Lambda,\ t>0; \quad \mathbf{p}^\varepsilon(0)=\mathbf{p}_0. \end{cases} \tag{9}$$

From now on, our goal is to "capture" *steady state solutions* of (9); to do so, we shall *time discretize* (9) by an *operator-splitting scheme à la Marchuk–Yanenko* (simply because it is the simplest splitting scheme we can think of; more sophisticated splitting schemes will be discussed elsewhere).

4. Time discretization of problem (9) by operator-splitting

Let Δt (>0) be a time discretization step; dropping the superscript ε we time-discretize (9) by the following *operator-splitting scheme* (of the *Marchuk–Yanenko's type*):

$$\mathbf{p}^0 = \mathbf{p}_0 \quad (= \mathbf{0},\ \textit{or}\ \nabla\varphi_1,\ \textit{for example}); \tag{10}$$

then, for $n \geqslant 0$, *assuming* $\mathbf{p}^n$ *known, solve successfully*:

$$\begin{cases} (\Delta t)^{-1}\displaystyle\int_\Omega \big(\mathbf{p}^{n+1/2}-\mathbf{p}^n\big)\cdot\mathbf{q}\,\mathrm{d}x + \int_\Omega \mathbf{p}^{n+1/2}\cdot\mathbf{q}\,\mathrm{d}x + \varepsilon_2^{-1}\sum_{i=1}^d\int_\Omega\big(|p_i^{n+1/2}|^2-1\big)p_i^{n+1/2}q_i\,\mathrm{d}x \\ \quad = C\displaystyle\int_\Omega\nabla\varphi_1\cdot\mathbf{q}\,\mathrm{d}x, \quad \forall\mathbf{q}\in\Lambda;\ \mathbf{p}^{n+1/2}\in\Lambda, \end{cases} \tag{11}$$

$$(\Delta t)^{-1}\int_\Omega\big(\mathbf{p}^{n+1}-\mathbf{p}^{n+1/2}\big)\cdot\mathbf{q}\,\mathrm{d}x + \big\langle\partial I_+(\mathbf{p}^{n+1}),\mathbf{q}\big\rangle = 0, \quad \forall\mathbf{q}\in\Lambda;\ \mathbf{p}^{n+1}\in\Lambda. \tag{12}$$

Problem (11) has a *unique* solution "as soon" as $\Delta t \leqslant \varepsilon_2$; moreover, (11) can be solved *pointwise* since, $\forall i = 1,\dots,d$, and a.e. on Ω, $p_i^{n+1/2}(x)$ is the solution of a *cubic equation in one variable* of the following type:

$$\big(1-\Delta t\varepsilon_2^{-1}+\Delta t\big)Z + \Delta t\varepsilon_2^{-1}Z^3 = \mathit{RHS}. \tag{13}$$

If $\varepsilon_2 \geqslant \Delta t$, (13) has a unique solution which can be easily solved by *Newton's method*. On the other hand, the solution of problem (12) is given by $\mathbf{p}^{n+1} = \nabla u^{n+1}$, u^{n+1} being the solution of the following *well-posed elliptic variational inequality*:

$$\begin{cases} \displaystyle\int_\Omega \nabla u^{n+1}\cdot\nabla\big(v-u^{n+1}\big)\,\mathrm{d}x + \Delta t\varepsilon_1\int_\Omega \Delta u^{n+1}\Delta\big(v-u^{n+1}\big)\,\mathrm{d}x \geqslant \int_\Omega \mathbf{p}^{n+1/2}\cdot\nabla\big(v-u^{n+1}\big)\,\mathrm{d}x, \\ \forall v\in H^2(\Omega)\cap K^+;\ u^{n+1}\in H^2(\Omega)\cap K^+. \end{cases} \tag{14}$$

In order to facilitate the numerical solution of problem (14) we perform a (variational) crime by *"approximately" factoring* the above problem as follows (other approximate factorizations are possible):

$$\int_\Omega \nabla\omega^{n+1}\cdot\nabla\big(v-\omega^{n+1}\big)\,\mathrm{d}x \geqslant \int_\Omega \mathbf{p}^{n+1/2}\cdot\nabla\big(v-\omega^{n+1}\big)\,\mathrm{d}x, \quad \forall v\in K^+;\ \omega^{n+1}\in K^+, \tag{15}$$

$$u^{n+1}-\Delta t\varepsilon_1\Delta u^{n+1}=\omega^{n+1} \quad \text{in } \Omega, \qquad u^{n+1}=0 \quad \text{on } \Gamma, \tag{16}$$

to be completed by

$$\mathbf{p}^{n+1}=\nabla u^{n+1}. \tag{17}$$

From the *non-negativity* of ω^{n+1}, the *maximum principle for second order elliptic operators* implies (via (16)) the *non-negativity* of u^{n+1}.

Remark 4.1. There is no basic difficulty at solving (14) numerically, but the approximate factorization (15), (16) of (14) produces smooth non-negative solutions as (14) does, while leading to faster algorithms. Indeed, $u^{n+1}\in H^3(\Omega)\cap K^+$ if Γ is smooth enough.

5. Finite element implementation of (10), (11), (15)–(17)

Since any solution of problem (1) is *continuous* and *piecewise affine*, it makes sense to compute these solutions via *globally continuous, piecewise affine finite element approximations.* From now on, we shall assume that Ω is a bounded *polygonal domain* of $\mathbb{R}^2$ (the non-polygonal case is almost as easy to treat). Let $\mathcal{T}_h$ be a finite element triangulation of $\overline{\Omega}$. We approximate $L^2(\Omega)$ (and $H^1(\Omega)$), $H_0^1(\Omega)$, K^+, Λ by

$$V_h=\big\{v_h \mid v_h\in C^0\big(\overline{\Omega}\big),\ v_h|_T\in P_1,\ \forall T\in\mathcal{T}_h\big\}, \tag{18}$$

$$V_{0h}=V_h\cap H_0^1(\Omega)=\big\{v_h \mid v_h\in V_h,\ v_h=0 \text{ on } \Gamma\big\}, \tag{19}$$

$$K_h^+=\big\{v_h \mid v_h\in V_{0h},\ v_h(P)\geqslant 0,\ \forall P \text{ vertex of } \mathcal{T}_h\big\}, \tag{20}$$

$$\Lambda_h=\big\{\mathbf{q}_h \mid \mathbf{q}_h\in\Lambda,\ \mathbf{q}_h|_T\in\mathbb{R}^2,\ \forall T\in\mathcal{T}_h\big\}, \tag{21}$$

respectively, with P_1 = space of the polynomials in x_1, x_2 of degree $\leqslant 1$. We clearly have $\nabla V_h\subset\Lambda_h$ and $\dim(\Lambda_h)=2\,\mathrm{Card}(\mathcal{T}_h)$. A *finite element implementation* of scheme (10), (11), (15)–(17), corresponding to (18)–(21), reads as follows (with $m(T)=\int_T \mathrm{d}x$, and other obvious notation):

$$\mathbf{p}_h^0=\mathbf{p}_{0h}, \quad \textit{with } \mathbf{p}_{0h} \textit{ given in } \Lambda_h; \tag{22}$$

for $n\geqslant 0$, $\mathbf{p}_h^n$ *being known, compute* $\mathbf{p}_h^{n+1/2}$, ω_h^{n+1}, u_h^{n+1} *and* $\mathbf{p}_h^{n+1}$ *as follows*:

$$\begin{cases} \dfrac{p_{iT}^{n+1/2}-p_{iT}^n}{\Delta t}+p_{iT}^{n+1/2}+\varepsilon_2^{-1}\big(\big|p_{iT}^{n+1/2}\big|^2-1\big)p_{iT}^{n+1/2}=\dfrac{C}{m(T)}\displaystyle\int_T\frac{\partial\varphi_1}{\partial x_2}\,\mathrm{d}x, \\ \forall i=1,2,\ \forall T\in\mathcal{T}_h, \end{cases} \tag{23}$$

$$\omega_h^{n+1}\in K_h^+;\ \forall v_h\in K_h^+, \quad \int_\Omega\nabla\omega_h^{n+1}\cdot\nabla\big(v_h-\omega_h^{n+1}\big)\,\mathrm{d}x\geqslant\int_\Omega\mathbf{p}_h^{n+1/2}\cdot\nabla\big(v_h-\omega_h^{n+1/2}\big)\,\mathrm{d}x, \tag{24}$$

$$\begin{cases} \mathbf{p}_h^{n+1}=\nabla u_h^{n+1},\ \textit{with } u_h^{n+1}\in V_{0h},\ \textit{and } \forall v_h\in V_{0h}, \\ \displaystyle\int_\Omega\big(u_h^{n+1}v_h+\Delta t\varepsilon_1\nabla u_h^{n+1}\cdot\nabla v_h\big)\,\mathrm{d}x=\int_\Omega\omega_h^{n+1}v_h\,\mathrm{d}x. \end{cases} \tag{25}$$

The cubic equations in (23) are particular cases of (13); they have a unique solution if $\Delta t\leqslant\varepsilon_2$, and can be solved easily by Newton's method. The *discrete obstacle problem* (24) is pretty classical; we found quite convenient to solve it by the penalty/Newton/conjugate gradient algorithm discussed in the companion note [5]. Finally, problem (25) is a fairly standard one (see, e.g., [3, Appendix 1] and the references therein) and does not deserve further comments other than: a sufficient condition to have a *discrete maximum principle* is to have the angles of $\mathcal{T}_h \leqslant \pi/2$ (a classical result indeed; see e.g., [2]). We have thus reduced the (approximate) solution of problem (1) to that of a sequence of simple cubic equations and of, essentially, *discrete linear elliptic problems.*

Remark 5.1. Let us denote by $\{u_h^\varepsilon\}_h^\varepsilon$ the family of approximate solutions provided by algorithm (18)–(25). The role of the *biharmonic regularization* (à la Tychonoff) provided by (24), (25), is to enhance the *compactness* properties of $\{u_h^\varepsilon\}_h^\varepsilon$ and (both things being related) control spatial oscillations of wave length of the order of h.

Remark 5.2. As can be expected, *biharmonic regularizations* are not new; they have been used in, e.g., [7] for the solution of boundary control problems for the wave equation, and in [9] for the solution of problems such as (1), precisely (by methods quite different from those discussed in this Note). Indeed, on the basis of *boundary layer thickness considerations* (similar to those in [7]), we anticipated that the optimal value of $\Delta t\varepsilon_1$ had to be of the order of h^2; numerical experiments validated this prediction, showing that a near optimal value for $\Delta t\varepsilon_1$ was $h^2/36$, for all h's sufficiently small.

Remark 5.3. With minor modifications, the methodology discussed above can be (and has been) applied to those variants of (1) where the boundary condition $u = 0$ on Γ is replaced by $u = g$, with g a Lipschitz continuous function such that $|g(x) - g(y)| \leqslant |x - y|, \forall x, y \in \Gamma$.

Remark 5.4. The methodology discussed above has been successfully applied to the solution of a "genuine" Eikonal equation, namely $\|\nabla u\| = 1$ a.e. in Ω, $u = g$ on $\partial\Omega$, with g as in Remark 5.3.

6. Numerical experiments

For the test problems discussed below we took $\Omega = (0, 1) \times (0, 1)$ and used a uniform triangulation of Ω. From a computational point of view, this implies that the finite element approximations reduce to finite difference ones, allowing, for example, the use of *cyclic reduction based* fast Poisson and Helmholtz solvers for the solution of the discrete elliptic problems encountered at each iteration of the numerical procedure discussed in the above sections. The main goal of the first and second test problems is to validate the numerical methodology since both of them have closed form solutions. The third test problem is the one of interest since the boundary conditions being incompatible with the constraints $|\partial u/\partial x_1| = |\partial u/\partial x_2| = 1$ a.e., we can expect a non-smooth behavior of the solutions (*fractal*, in fact) near the boundary.

First two test problems: They are defined by

$$\left|\frac{\partial u}{\partial x_1}\right| = \left|\frac{\partial u}{\partial x_2}\right| = 1 \quad \text{a.e. in } \Omega, \qquad u = g \quad \text{on } \Gamma, \tag{26}$$

with

$$\begin{cases} g(x_1, 0) = g(x_1, 1) = \min(x_1, 1 - x_1), & 0 \leqslant x_1 \leqslant 1, \\ g(0, x_2) = g(1, x_2) = \min(x_2, 1 - x_2), & 0 \leqslant x_2 \leqslant 1. \end{cases}$$

Here, the maximal (which turns out to be also a viscosity) solution of problem (26) is given by

$$u_{\max}(x_1, x_2) = 1 - \left|\frac{1}{2} - x_1\right| - \left|\frac{1}{2} - x_2\right| = \min(x_1, 1 - x_1) + \min(x_2, 1 - x_2).$$

The minimal solution is given by

$$u_{\min}(x_1, x_2) = \begin{cases} |x_1 - x_2|, & 0 \leqslant x_1, x_2 \leqslant 0.5, \text{ and } 0.5 \leqslant x_1, x_2 \leqslant 1, \\ |1 - x_1 - x_2|, & 0.5 \leqslant x_1 \leqslant 1,\ 0 \leqslant x_2 \leqslant 0.5, \text{ and } 0 \leqslant x_1 \leqslant 0.5,\ 0.5 \leqslant x_2 \leqslant 1. \end{cases}$$

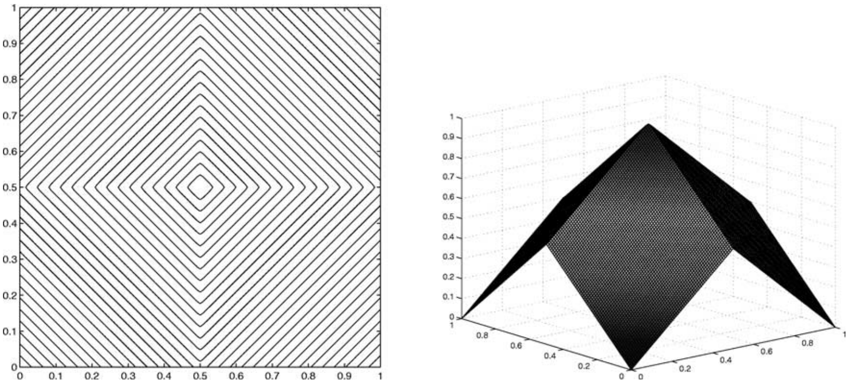

Fig. 1. Contours (left) and graph (right) of the computed maximal solution of (26) ($h = 1/128$).

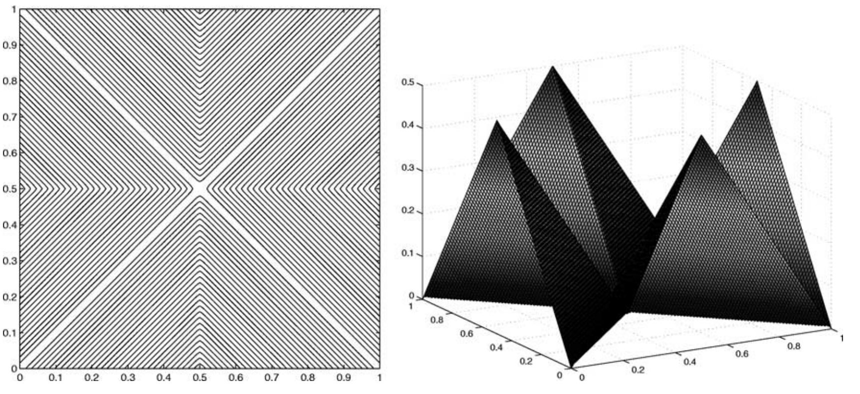

Fig. 2. Contours (left) and graph (right) of the computed minimal solution of (26) ($h = 1/128$).

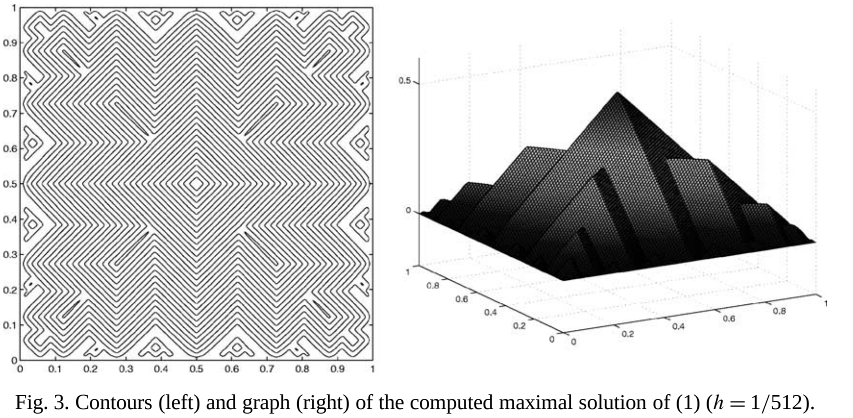

Fig. 3. Contours (left) and graph (right) of the computed maximal solution of (1) ($h = 1/512$).

The computed solution u_h obtained with $h = 1/128$, $\Delta t \varepsilon_1 = h^2/36$, $\varepsilon_2 = 0.001$, $\Delta t = 0.0001$, and $C = 10$ for the maximal case and $C = -10$ for the minimal case, has been visualized in Figs. 1 and 2, respectively; it coincides quite well with the exact solution since approximation errors are $\|u_h - u\|_{0,\Omega} = 3.268 \times 10^{-4}$ and $\|u_h - u\|_{\infty,\Omega} = 5.086 \times 10^{-3}$ (and $\|u_h - u\|_{0,\Omega} = 7.133 \times 10^{-4}$ and $\|u_h - u\|_{\infty,\Omega} = 5.715 \times 10^{-3}$, respectively).

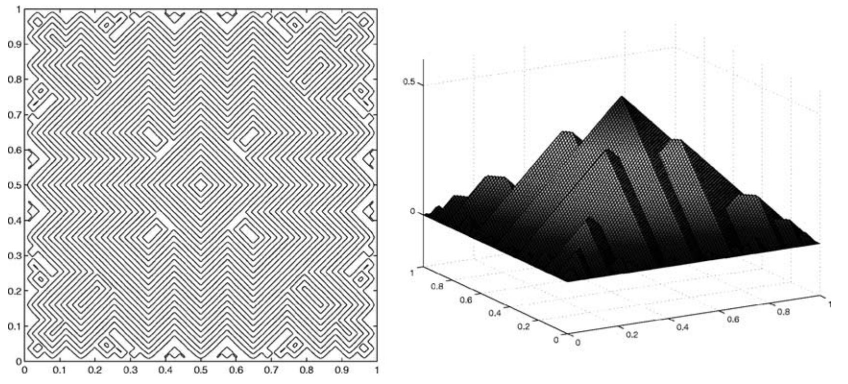

Fig. 4. Contours (left) and graph (right) of the computed maximal solution of (1) ($h = 1/1024$).

Third test problem: In this case we look for a nearly maximal solution of problem (1), recalling that in this case maximal solutions do not exist. We have visualized in Figs. 3 and 4 the graphs and contours of the discrete solutions computed with $h = 1/512$ and $1/1024$. The other parameters are $\Delta t\varepsilon_1 = h^2/9$, $\varepsilon_2 = 0.001$, $C = 10$ and $\Delta t = 0.0001$. These figures show that as one refines the mesh new structures appear clearly showing (as expected) a fractal behavior near the boundary (and particularly in the corners). Incidentally, the L^1-norms corresponding to $h = 1/512$ and $h = 1/1024$ are 0.142263 and 0.143316, respectively, while the maximal values (reached at $x_1 = x_2 = 1/2$) are 0.52457 and 0.51494 (they should be equal to 0.5, but it should be realized that we are solving a highly non-smooth problem).

For this test problem, it can be shown that the maximizing sequences associated to (2) converges strongly in $C^0(\overline{\Omega})$ and weakly in $H_0^1(\Omega)$ to $\tilde{u} : x \to \mathit{distance}(x, \Gamma)$. We clearly have $\tilde{u} \notin E$ and $\int_\Omega \tilde{u}\,\mathrm{d}x = 1/6\ (= 0.16666\ldots)$.

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