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Probability Theory

Application of large deviation methods to the pricing of index options in finance

Méthodes de grandes déviations et pricing d'options sur indice

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Abstract

We develop an asymptotic formula for calculating the implied volatility of European index options based on the volatility skews of the options on the underlying stocks and on a given correlation matrix for the basket. The derivation uses the steepest-descent approximation for evaluating the multivariate probability distribution function for stock prices, which is based on large-deviation estimates of diffusion processes densities by Varadhan (Comm. Pure Appl. Math. 20 (1967)). A detailed version of these results can be found in (RISK 15 (10) (2002)). *To cite this article: M. Avellaneda et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

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Résumé

Nous montrons une formule asymptotique donnant la volatilité implicite d'une option sur indice à partir des volatilités des actifs sous-jacents. La démonstration repose sur les estimations de densités de diffusion en temps petit du type grandes déviation de Varadhan (Comm. Pure Appl. Math. 20 (1967)). On pourra trouver une version détaillée de ces résultats dans l'article (RISK 15 (10) (2002)). *Pour citer cet article : M. Avellaneda et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

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1. Introduction

We consider a basket of n stocks described by their price processes $S_i(t)$, $i = 1, \dots, n$, and an index on these stocks $B(t) = \sum_{i=1}^n w_i S_i(t)$, with the w_i 's constant.

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For simplicity we assume that each stock follows a one-factor risk-neutral process $dS_i/S_i = \sigma_i(S_i, t) dZ_i + \mu_i dt$, where $\sigma_i(S_i, t)$ is the so-called *local volatility function*, μ_i is the drift associated with the cost of carry, and $Z_i = Z_i(t)$ are standard Brownian motions which satisfy $\mathbb{E}(dZ_i dZ_j) = \rho_{ij} dt$, ρ_{ij} constant.¹ Typically, local volatility functions for a particular underlying asset are derived from option market quotes by well-known methods [5–7,10].

To obtain the fair value of an index option in relation to its components, we compute an effective local volatility function for the index, $\sigma_{B,\text{loc}}(B, t)$. This function is consistent with information on individual stock options and stock correlations, and has the property that the equation

$$\frac{dB}{B} = \sigma_{B,\text{loc}}(B, t) dW + \mu_B dt \tag{1}$$

describes the evolution of the index the a “risk-neutral” world. (Here μ_B is the effective cost-of-carry rate for the index.) Accordingly, the price of an European call option with maturity t and strike B is given by $\mathbb{E}(e^{-rt}(B(t) - B)_+)$, where $B(t)$ follows (1) and r is the interest rate.

From general principles [4,6,8,9], the square of the local volatility function in (1) is found to be the conditional expectation of the *stochastic variance* $\sigma_B^2(\mathbf{S}, t) = \frac{1}{B^2} \sum_{i,j=1}^n \rho_{ij} \sigma_i \sigma_j w_i w_j S_i S_j$ given the value of the index B at time t . More precisely, we have:

$$\sigma_{B,\text{loc}}^2(B, t) = \mathbb{E}\{\sigma_B^2 \mid B(t) = B\} = \mathbb{E}\left\{ \frac{1}{B^2} \sum_{i,j=1}^n \rho_{ij} \sigma_i \sigma_j w_i w_j S_i(t) S_j(t) \mid \sum_{i=1}^n w_i S_i(t) = B \right\}. \tag{2}$$

In this paper, we propose an approximation to the function $\sigma_{B,\text{loc}}^2(B, t)$. This approximation gives rise to a formula that links the option prices in individual stocks, the correlations between different stocks, and the values of options on indices.

Let us note for further use that an important way of representing option prices in practice is by means of the so-called Black-Scholes *implied volatility*: one replaces $\sigma_{B,\text{loc}}$ in (1) by a *constant* σ_B^I so as to leave the value of the option maturing at t with strike B unchanged. Note, however, that this constant depends on strike and maturity as parameters. One proceeds in a similar way to define the σ_i^I 's.

Our strategy is to establish an asymptotic relation between the local and implied volatilities of the index and the underlying assets in the limit $\bar{\sigma}^2 t \ll 1$, where $\bar{\sigma}$ denotes a typical level of the volatilities involved. A typical order of magnitude for the dimensionless parameter $\bar{\sigma}^2 t$ for major equity indices is 10^{-2} , which is small enough to justify the use of asymptotic methods. In practice, we found that the volatility reconstruction formulas described hereafter are in excellent agreement with contemporaneous market quotes [1].

2. Main results

It is convenient to introduce the forward spot prices $F_i = S_i(0) e^{\mu_i t}$, $F = B(0) e^{\mu_B t}$, the forward log-moneyness $x^i = \ln(S_i/F_i)$, $\bar{x} = \ln(B/F)$ and the fractions of holdings $p_i(\mathbf{x}) = (F_i e^{x_i} w_i) / (\sum_{k=1}^n F_k e^{x_k} w_k)$. Slightly abusing the notations, we simply write $\sigma_i(x^i) = \sigma_i(F_i e^{x^i}, 0)$ and $\sigma_{B,\text{loc}}(\bar{x}) = \sigma_{B,\text{loc}}(F e^{\bar{x}}, 0)$ (and similarly for σ_i^I, σ_B^I).

Theorem 2.1 (Link between local volatilities). *In the limit $\bar{\sigma}^2 t \ll 1$ the local volatility of the index is given, to first order, by*

$$\sigma_{B,\text{loc}}^2(\bar{x}) = \sum_{i,j=1}^n \rho_{ij} \sigma_i(x_i^*) \sigma_j(x_j^*) p_i(\mathbf{x}^*) p_j(\mathbf{x}^*), \tag{3}$$

¹ The results presented here apply to more general correlation/volatility structures, including for instance the case of multivariate stochastic volatility/stochastic correlation models.

where $x^* = (x_1^*, \dots, x_n^*)$ is the solution of the nonlinear system

$$\int_0^{x_i^*} \frac{du}{\sigma_i(u)} = \lambda \sum_{j=1}^n \rho_{ij} p_j(x^*) \sigma_j(x_j^*), \quad \forall i = 1, \dots, n; \quad \sum_{i=1}^n w_i F_i e^{x_i^*} = B. \tag{4}$$

Theorem 2.2 (Link between Black–Scholes implied volatilities).

(i) In the limit $\bar{\sigma}^2 t \ll 1$ the implied volatility functions of the index and the underlying stocks are related by

$$\sigma_B^I(\bar{x}) = \left(\frac{1}{\bar{x}} \int_0^{\bar{x}} \frac{du}{\sigma_{B,\text{loc}}(u)} \right)^{-1}, \quad \sigma_i(x^i) = \left(\frac{d}{dy} \left(\frac{y}{\sigma_i^I(y)} \right) \Big|_{y=x^i} \right)^{-1}, \tag{5}$$

($i = 1, \dots, n$) together with (3), (4).

(ii) In the at-the-(forward)money region $\{|\bar{x}| \ll 1, |x_i| \ll 1\}$ this relation reduces, to first order, to

$$2\sigma_B^I(\bar{x}) - \sigma_B^I(0) = \sqrt{\sum_{i,j=1}^n \rho_{ij} p_i(\mathbf{x}^*) p_j(\mathbf{x}^*) (2\sigma_i^I(x_i^*) - \sigma_i^I(0)) (2\sigma_j^I(x_j^*) - \sigma_j^I(0))}. \tag{6}$$

(iii) The most likely configuration corresponding to a given index displacement is characterized, to first order, by

$$\frac{x_i^*}{\sigma_i^I(0)} \simeq \frac{\bar{x}}{\sigma_B^I(0)} \left(\sum_{j=1}^n \rho_{ij} p_j(\mathbf{0}) \frac{\sigma_j^I(0)}{\sigma_B^I(0)} \right). \tag{7}$$

3. Sketch of the proof

3.1. Proof of Theorem 2.1

We formally rewrite (2) as

$$\sigma_{B,\text{loc}}^2 = \frac{\mathbb{E}\{\sigma_B^2 \delta(B(t) - B)\}}{\mathbb{E}\{\delta(B(t) - B)\}} \tag{8}$$

so that $\sigma_{B,\text{loc}}^2$ appears as an average of σ_B^2 . When $\bar{\sigma}^2 t$ is small, we shall prove that a concentration phenomenon appears, reducing (8) to an evaluation at some point. For this purpose, we introduce the transition probability, or Green function, $\pi(\mathbf{0}, 0; \mathbf{x}, t)$ of the diffusion process $\mathbf{x} = (x^1, \dots, x^n)$ with matrix $a^{ij} = \sigma_i \sigma_j \rho_{ij}$ and we analyze it thanks to a classical formula by Varadhan [11] that we now recall. Introducing the inverse matrix $(g_{ij}) = (a^{ij})^{-1}$ and the associated Riemmanian metric $ds^2 = \sum_{i,j=1}^n g_{ij} dx^i dx^j$, we have

$$\pi(\mathbf{0}, 0; \mathbf{x}, t) \sim e^{-d^2(\mathbf{0},\mathbf{x})/(2t)} = e^{-(\bar{\sigma})^2 d^2(\mathbf{0},\mathbf{x})/(2(\bar{\sigma})^2 t)}, \tag{9}$$

where

$$d^2(\mathbf{0}, \mathbf{x}) = \inf_{\mathbf{x}(0)=0, \mathbf{x}(1)=\mathbf{x}} \int_0^1 \sum_{i,j=1}^n g_{ij}(\mathbf{x}(\tau), 0) \dot{x}^i \dot{x}^j d\tau. \tag{10}$$

Here $\dot{\mathbf{x}}$ is the time-derivative of \mathbf{x} . The asymptotics in (9) are understood in the sense that the ratio of the logarithms of the two terms tends to 1 as $\bar{\sigma}^2 t \rightarrow 0$.

Hence $\pi(\mathbf{0}, 0; \mathbf{x}, t)$ is strongly peaked near the points \mathbf{x} where $d^2(\mathbf{0}, \mathbf{x})$ is minimal. The method of steepest descent thus implies $\sigma_{B, \text{loc}}(\bar{x}, t) \simeq \sigma_B(\mathbf{x}^*, t)$, where \mathbf{x}^* is the (generically unique) point realizing the distance (in the sense of (10)) of the origin to the manifold $\Gamma_B = \{\mathbf{x}: \sum_{i=1}^n w_i F_i e^{x^i} = B\}$.

Introducing the change of variable $y^i = \int_0^{x^i} du/\sigma_i(u)$, determining \mathbf{x}^* is easily shown to be equivalent to minimizing $\int_0^1 \sum_{i,j=1}^n (\rho^{-1})_{ij} \dot{y}^i \dot{y}^j d\tau$ under the constraint $\sum_{i=1}^n w_i F_i(0) e^{x^i(y^i)} = B$. Writing the first-order Euler–Lagrange condition results in

$$\sum_{j=1}^n (\rho^{-1})_{ij} \dot{y}^j = \frac{\lambda}{B} w_i F_i(0) e^{x^i(y^i)} \frac{\partial x^i(y^i)}{\partial y^i} = \lambda p_i(x^i(y^i)) \sigma_i(x^i(y^i)),$$

where for convenience the Lagrange multiplier has been written as λ/B . Looking for a solution $\mathbf{y}(\tau)$ linear in τ (so that $\dot{y}^j = y^j$) and multiplying by $(\rho)^{-1}$ easily yields (4).

3.2. Proof of Theorem 2.2

Part 1 follows readily from the harmonic-mean relation between implied and local volatilities in the limit $\bar{\sigma}^2 t \ll 1$ which can be found in [2,3]. Part 2 follows from a first-order expansion of that relation near $\bar{x} = 0$ (resp. $\bar{x}_i = 0$). Similarly, Part 3 follows by Taylor expansion using Eqs. (4) and the Euler–Lagrange optimality conditions for the vector \mathbf{y} .

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