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Weak convergence results for inhomogeneous rotating fluid equations

Résultats de convergence faible pour des équations des fluides tournants non homogènes

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Abstract

We consider the equations governing incompressible, viscous fluids in three space dimensions, rotating around an inhomogeneous vector $B(x)$: this is a generalization of the usual rotating fluid model (where B is constant). We prove the weak convergence of Leray-type solutions towards a vector field which satisfies the usual 2D Navier–Stokes equation in the regions of space where B is constant, with Dirichlet boundary conditions, and a heat–type equation elsewhere. The method of proof uses weak compactness arguments. ***To cite this article: I. Gallagher, L. Saint-Raymond, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

On considère les équations modélisant des fluides incompressibles et visqueux en trois dimensions d'espace, en rotation rapide autour d'un vecteur non homogène $B(x)$: on généralise ainsi le modèle habituel des fluides tournants (où B est constant). On montre la convergence des solutions de Leray vers un champ de vecteurs qui vérifie les équations habituelles de Navier–Stokes 2D dans les régions de l'espace où B est constant, avec des conditions aux limites de Dirichlet, et une équation de type chaleur ailleurs. La méthode de démonstration repose sur des arguments de compacité faible. ***Pour citer cet article : I. Gallagher, L. Saint-Raymond, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

Dans cette Note nous présentons des résultats concernant le comportement asymptotique des solutions des équations des fluides tournants dans le cas où le vecteur de rotation n'est pas homogène. Nous renvoyons par exemple aux articles [1–3,5] pour des études dans le cas constant.

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On se place donc dans un domaine $\Omega = \Omega_h \times \Omega_3$, où Ω_h est soit l'espace entier $\mathbf{R}^2$ soit un domaine périodique de $\mathbf{R}^2$, et de même Ω_3 est $\mathbf{R}$ ou $\mathbf{T}$. Et on considère le système (1) des équations de Navier–Stokes des fluides incompressibles, pénalisé par un terme de rotation $\frac{1}{\varepsilon} u \wedge B$ où $B = be_3$ et b est une fonction régulière sur Ω_h qui ne s'annule pas. Il n'est pas difficile de voir qu'il existe des solutions à la Leray à ce système (voir [4]), bornées uniformément en ε dans l'espace $L^\infty(\mathbf{R}^+, L^2(\Omega)) \cap L^2(\mathbf{R}^+, \dot{H}^1(\Omega))$; on peut donc extraire une limite faible $\bar{u}$, et notre objectif est de décrire le comportement d'une telle limite quand ε tend vers zéro. Ce comportement est bien sûr relié à la structure de l'opérateur de pénalisation singulière $L : u \in \mathrm{H} \mapsto P(u \wedge B) \in \mathrm{H}$ où P est le projecteur de Leray de $L^2(\Omega)$ sur son sous-espace H de champs de vecteurs de divergence nulle. Il semble néanmoins difficile d'étudier le spectre de L car il est à coefficients variables. Nous n'allons donc pas ici utiliser les méthodes habituelles du cas où b est constant, mais se tourner plutôt vers une « méthode de compacité faible » dans l'esprit de Lions et Masmoudi [6] (pour la limite incompressible).

Avant d'énoncer le résultat que nous voulons démontrer, donnons quelques définitions. On notera $S \stackrel{\text{def}}{=} \{x \in \Omega \mid \nabla b(x) = 0\}$ et $\mathcal{O} \stackrel{\text{def}}{=} \{x \in \Omega \mid \nabla b(x) \neq 0\}$ et $\mathcal{S}$ l'intérieur de S. Pour simplifier nous supposerons que l'ensemble $\Omega \setminus (\mathcal{O} \cup \mathcal{S})$ est de mesure de Lebesgue nulle (H0), que $\mathcal{S}$ est un domaine régulier (H1), et que sur chaque composante connexe $\mathcal{O}_j$ de $\mathcal{O}$, il y a une fonction régulière σ_j telle que (b, σ_j, x_3) est un système de coordonnées globales régulières (H2) :

$$\mathcal{O}_j = \{x_h \in \mathbf{R}^2 \mid (b(x_h), \sigma_j(x_h)) \in B_j \times \Sigma_j\} \times \Omega_3.$$

Théorème 0.1. *Supposons que $B = be_3$ où $b = b(x_h)$ est une fonction strictement positive et régulière, par exemple une perturbation $C_c^\infty(\Omega)$ d'une constante, et telle que les hypothèses* (H0) *à* (H2) *sont vérifiées.*

Soit u^0 un champ de vecteurs de divergence nulle dans $L^2(\Omega)$; et pour tout $\varepsilon > 0$ soit u_ε une solution faible de (1) *de donnée initiale u^0. Alors u_ε converge faiblement dans $L^2_{\text{loc}}(\mathbf{R}^+ \times \Omega)$ vers une limite $\bar{u}$.*

Si $\Omega_3 = \mathbf{R}$, $\bar{u} = 0$. Si $\Omega_3 = \mathbf{T}$, $\bar{u}$ est décrite par le système d'équations suivant : la troisième composante $\bar{u}_3$ vérifie l'équation de transport (4) *sur $\mathbf{R}^+ \times \Omega$; la composante horizontale $\bar{u}_h$ vérifie les équations de Navier–Stokes bidimensionnelles avec conditions de Dirichlet au bord sur $\mathbf{R}^+ \times \mathcal{S}$, et une équation de type chaleur* (5) *sur $\mathbf{R}^+ \times \mathcal{O}$, où Π est la projection orthogonale de $L^2(\Omega)$ sur* $\mathrm{Ker}(L)$ *(qui peut être étendue à $\mathcal{D}'(\mathcal{O} \cup \mathcal{S})$).*

Donnons une idée de la démonstration de ce résultat ; nous renvoyons à [4] pour les arguments précis.

La première étape consiste à étudier le noyau de l'opérateur L. On montre par un calcul algébrique qu'il est formé des champs de vecteurs u qui ne dépendent que de deux variables, de la forme $u = \nabla_h^\perp \varphi + \alpha e_3$ avec $\nabla_h \varphi \in L^2(\Omega_h)$, $\alpha \in L^2(\Omega_h)$ et $\nabla_h b \cdot \nabla_h^\perp \varphi = 0$. Il n'est pas très difficile de se convaincre que $\bar{u}$ appartient à $\mathrm{Ker}(L)$. Dans le cas $\Omega_3 = \mathbf{R}$ on en déduit immédiatement que la limite est nulle.

L'idée est ensuite de décomposer $u_\varepsilon = \bar{u}_\varepsilon + w_\varepsilon$, où $\bar{u}_\varepsilon = \Pi u_\varepsilon$ est la projection de u_ε sur $\mathrm{Ker}(L)$. Comme Π est continue sur $H^s(\mathcal{O} \cup \mathcal{S})$ (sous les hypothèse (H0)–(H2)), on peut alors montrer que $\bar{u}_\varepsilon = \Pi u_\varepsilon$ est compacte dans $L^2_{\text{loc}}(\mathbf{R}^+ \times \Omega)$, et que w_ε converge faiblement vers zéro dans $L^2_{\text{loc}}(\mathbf{R}^+ \times \Omega)$.

On obtient donc que $\Pi(\bar{u}_\varepsilon \cdot \nabla \bar{u}_\varepsilon) \rightharpoonup \Pi(\bar{u} \cdot \nabla \bar{u})$, et $\Pi(w_\varepsilon \cdot \nabla \bar{u}_\varepsilon + \bar{u}_\varepsilon \cdot \nabla w_\varepsilon) \rightharpoonup 0$ dans $\mathcal{D}'(\mathbf{R}^+ \times (\mathcal{O} \cup \mathcal{S}))$, ainsi que le fait que $\Pi \Delta u_\varepsilon \rightharpoonup \Pi \Delta \bar{u}$, et $\partial_t \Pi u_\varepsilon \rightharpoonup \partial_t \bar{u}$. La difficulté consiste à montrer que $\Pi(w_\varepsilon \cdot \nabla w_\varepsilon)$ converge faiblement vers zéro dans $\mathcal{D}'(\mathbf{R}^+ \times (\mathcal{O} \cup \mathcal{S}))$. C'est à cet endroit que les méthodes de compacité faible (par compensation) s'avèrent opérantes et permettent de montrer qu'il n'y a pas de contribution oscillante dans le système limite.

On identifie enfin l'équation de la chaleur en observant simplement que

$$\forall \Phi \in \mathrm{Ker}\, L \cap C_c^\infty(\mathcal{O}), \quad (\bar{u}_h \cdot \nabla_h \bar{u}_h | \Phi_h)_{L^2} = 0.$$

Ce dernier résultat s'obtient par la forme particulière du noyau de L présenté ci-dessus.

1. Introduction

In this Note we present results concerning the asymptotics of solutions of rotating fluid equations, in the case where the rotation vector is non homogeneous. We consider a domain $\Omega = \Omega_h \times \Omega_3$, where Ω_h denotes either the whole space $\mathbf{R}^2$ or any periodic domain of $\mathbf{R}^2$, and similarly Ω_3 denotes $\mathbf{R}$ or $\mathbf{T}$. We are interested in the following system:

$$\begin{aligned} &\partial_t u + u \cdot \nabla u - \nu \Delta u + \frac{1}{\varepsilon} u \wedge B + \nabla p = 0 \quad \text{on } \mathbf{R}^+ \times \Omega, \\ &\nabla \cdot u = 0 \quad \text{on } \mathbf{R}^+ \times \Omega, \qquad u_{|t=0} = u^0 \quad \text{on } \Omega, \end{aligned} \tag{1}$$

where $B = be_3$ is the rotation vector, and b is a smooth function defined in Ω_h which does not vanish.

The rotating fluid equations, with b constant and homogeneous, modelize the movement of the atmosphere or the oceans at mid-latitudes. The fluid is supposed to be incompressible and its viscosity is $\nu > 0$. The vector field u is the velocity and the scalar p is the pressure, both are unknown. The parameter ε is the Rossby number, and its inverse stands for the speed of rotation of the Earth. Note that one can also see B as a magnetic field, in which case it makes sense to understand what happens when b is not homogeneous; that also holds if one wants to study the movement of the atmosphere in other regions than mid-latitudes.

In the constant case, those equations have been studied by a number of authors. We refer for instance to the works of Babin et al. [1], Gallagher [3], Grenier [5] for the periodic case, and Chemin et al. [2] for the whole space case. The results in those papers concern both weak and strong solutions; in this Note we shall only be concerned with Leray-type weak solutions, whose existence is easy to prove: one can show (see [4]) that if the initial data u^0 is a divergence free vector field in L^2, then for all $\varepsilon > 0$, Eq. (1) has at least one weak solution $u_\varepsilon \in L^\infty(\mathbf{R}^+, L^2(\Omega)) \cap L^2(\mathbf{R}^+, \dot{H}^1(\Omega))$, which moreover satisfies the following energy estimate:

$$\left\| u_\varepsilon(t) \right\|_{L^2}^2 + 2\nu \int_0^t \left\| \nabla u_\varepsilon(s) \right\|_{L^2}^2 \mathrm{d}s \leqslant \left\| u^0 \right\|_{L^2}^2. \tag{2}$$

Since we have a bounded family of solutions, one can construct a weak limit point $\bar{u}$, and the question we want to address is to find the equation satisfied by $\bar{u}$. Of course the problem consists in taking the limit in the non linear part of the equation. Let us define the operator

$$L : u \in \mathrm{H} \mapsto P(u \wedge B) \in \mathrm{H}, \tag{3}$$

where P denotes the Leray projection from $L^2(\Omega)$ onto its subspace H of divergence-free vector fields. The study of the spectrum of L seems to be a difficult issue due to the fact that it has variable coefficients; so we do not want to apply the usual, constant b methods, as to our knowledge they all involve spectral properties of L. The idea therefore is to turn to what is known as "weak compactness methods", in the spirit of Lions and Masmoudi [6] (for the incompressible limit). Before stating the result, let us give some definitions. We will denote by $S \stackrel{\text{def}}{=} \{x \in \Omega \mid \nabla b(x) = 0\}$ and $\mathcal{O} \stackrel{\text{def}}{=} \{x \in \Omega \mid \nabla b(x) \neq 0\}$ and by $\mathcal{S}$ the interior of S. For the sake of simplicity, we will assume in the sequel that

(H0) the set $\Omega \setminus (\mathcal{O} \cup \mathcal{S})$ is of Lebesgue measure 0,
(H1) $\mathcal{S}$ is a smooth domain,
(H2) on each connected component $\mathcal{O}_j$ of $\mathcal{O}$, there is a smooth function σ_j such that (b, σ_j, x_3) is a global smooth coordinate system and $\mathcal{O}_j = \{x_h \in \mathbf{R}^2 \mid (b(x_h), \sigma_j(x_h)) \in B_j \times \Sigma_j\} \times \Omega_3$.

Theorem 1.1. *Suppose that $B = be_3$ where $b = b(x_h)$ is a smooth nonnegative function, say a $C_c^\infty(\Omega)$ perturbation of a constant, and where assumptions* (H0)–(H2) *are satisfied. Let u^0 be any divergence free vector field in $L^2(\Omega)$, and for every $\varepsilon > 0$ let u_ε be any weak solution associated with u^0. Then u_ε converges weakly in $L^2_{\text{loc}}(\mathbf{R}^+ \times \Omega)$ to* *a limit $\bar{u}$ which if $\Omega_3 = \mathbf{R}$ is zero, and if $\Omega_3 = \mathbf{T}$ is defined as follows: the third component $\bar{u}_3$ satisfies the transport equation*

$$\partial_t \bar{u}_3 - \nu \Delta_h \bar{u}_3 + \bar{u}_h \cdot \nabla_h \bar{u}_3 = 0, \quad \partial_3 \bar{u}_3 = 0, \quad \bar{u}_{3|t=0} = \int_{\mathbf{T}} u_3^0(x_h, x_3)\,\mathrm{d}x_3 \quad \textit{in } \mathbf{R}^+ \times \Omega,$$

while the horizontal component $\bar{u}_h$ depends on the region of space considered:

- *in $\mathcal{S}$, $\bar{u}_h$ satisfies the two dimensional Navier–Stokes equations with Dirichlet boundary conditions*;
- *in $\mathcal{O}$, $\bar{u}_h$ satifies a heat-type equation*
$$\partial_t \bar{u}_h - \nu \Pi \Delta_h \bar{u}_h = 0, \tag{4}$$
where Π denotes the orthogonal projection from $L^2(\Omega)$ onto $\mathrm{Ker}(L)$ (*which can be extended to $\mathcal{D}'(\mathcal{O} \cup \mathcal{S})$*).

In the regions where b is homogeneous, we recover at the limit the 2D Navier–Stokes equations as usual. The Dirichlet boundary conditions appear quite naturally, considering that on the other side of the boundary one finds that $\bar{u}_h$ is proportional to $\nabla_h^\perp b$ which vanishes on the boundary of $\mathcal{S}$. More surprising is certainly the result in the region where b is not homogeneous: it can be understood as some sort of turbulent behaviour, where all scales are mixed due to the variation of b. Note that (4) is indeed a heat equation because we will see that $-(\Pi \Delta_h \bar{u}_h | \bar{u}_h)_{L^2(\Omega)} = \|\nabla_h \bar{u}_h\|_{L^2}^2$.

2. Proof of the theorem

We shall present here the main arguments leading to the result, and we refer to [4] for the precise proof. The first step consists in studying the operator L defined in (3). In particular, we can prove that $\bar{u}$ belongs to the kernel $\mathrm{Ker}(L)$ of L, which is characterized in the following proposition.

Proposition 2.1. *Define the linear operator L by* (3). *Then, $u \in \mathrm{H}$ belongs to* $\mathrm{Ker}(L)$ *if and only if there exist $\nabla_h \varphi \in L^2(\Omega_h)$ and $\alpha \in L^2(\Omega_h)$ with $\nabla_h b \cdot \nabla_h^\perp \varphi = 0$, such that $u = \nabla_h^\perp \varphi + \alpha e_3$.*

The proof of this result is left to the reader, as well as the proof of the fact that $\bar{u}$ belongs to $\mathrm{Ker}(L)$. Note that in the case $\Omega_3 = \mathbf{R}$ we infer that the limit is zero simply because it only depends on x_h and is in $L^2(\Omega)$. So from now on we can suppose that $\Omega_3 = \mathbf{T}$. An important point to note is also that the orthogonal projection Π is continuous on $H^s(\mathcal{O} \cup \mathcal{S})$ for all $s \in \mathbf{R}$ (where for all $s \geqslant 0$, $H^s(\mathcal{O} \cup \mathcal{S})$ is the closure of $C_c^\infty(\mathcal{O} \cup \mathcal{S})$ for the H^s norm, and for $s \geqslant 0$, $H^{-s}(\mathcal{O} \cup \mathcal{S})$ is the dual space of $H^s(\mathcal{O} \cup \mathcal{S})$). Contrary to the constant case, it is not necessarily continuous on $H^s(\Omega)$ for $s > \frac{1}{2}$, which implies in particular that the penalization does not preserve H^s norms [4].

The second step consists in considering separately $\bar{u}_\varepsilon \stackrel{\text{def}}{=} \Pi u_\varepsilon$ and $w_\varepsilon \stackrel{\text{def}}{=} u_\varepsilon - \bar{u}_\varepsilon$. It is not too difficult to prove that $(\bar{u}_\varepsilon)$ is strongly compact in $L^2_{\mathrm{loc}}(\mathbf{R}^+ \times \Omega)$, and that w_ε converges weakly towards zero in $L^2_{\mathrm{loc}}(\mathbf{R}^+ \times \Omega)$ (see [4]). From those results we infer in particular that $\Pi(\bar{u}_\varepsilon \cdot \nabla \bar{u}_\varepsilon) \rightharpoonup \Pi(\bar{u} \cdot \nabla \bar{u})$, and $\Pi(w_\varepsilon \cdot \nabla \bar{u}_\varepsilon + \bar{u}_\varepsilon \cdot \nabla w_\varepsilon) \rightharpoonup 0$ in $\mathcal{D}'(\mathbf{R}^+ \times (\mathcal{O} \cup \mathcal{S}))$. By the continuity of Π we know that $\Pi \Delta u_\varepsilon \rightharpoonup \Pi \Delta \bar{u}$, and $\partial_t \Pi u_\varepsilon \rightharpoonup \partial_t \bar{u}$. Now we need to find the limit of $\Pi(w_\varepsilon \cdot \nabla w_\varepsilon)$ and finally to identify $\bar{u}$. The main result we need to prove is the following.

Proposition 2.2. *Under the above assumptions, the following result holds*: *$\Pi(w_\varepsilon \cdot \nabla w_\varepsilon)$ goes weakly to zero in $\mathcal{D}'(\mathbf{R}^+ \times (\mathcal{O} \cup \mathcal{S}))$.*

To prove that result, we use weak compactness methods: we start by noticing that due to the form of Π, the third component $w_{\varepsilon,3}$ of w_ε can be written $w_{\varepsilon,3} = \partial_3 W_\varepsilon$ with $W_\varepsilon \in L^2_{\mathrm{loc}}(\mathbf{R}^+, \Omega)$. It is then easy to prove that

$$\varepsilon \partial_t (w_{\varepsilon,h} - \nabla_h W_\varepsilon) + w_{\varepsilon,h} \wedge B = r_{\varepsilon,h}, \qquad \varepsilon \partial_t (\partial_2 w_{\varepsilon,1} - \partial_1 w_{\varepsilon,2}) + \mathrm{div}_h (b w_{\varepsilon,h}) = r_{\varepsilon,3}, \tag{5}$$

where r_ε goes to zero in $L^2_{\text{loc}}(\mathbf{R}^+, H^{-5/2}_{\text{loc}}(\mathcal{O}\cup\mathcal{S}))$. The idea is then, as in [6], to smoothen out Eq. (5) by a convolution and to prove Proposition 2.2 first on the smoothened solutions. Then one proves that the difference between the two solutions goes strongly to zero; that second argument is left to the reader. From now on let us suppose that the remainder term r_ε goes strongly to zero in $L^2_{\text{loc}}(\mathbf{R}^+\times(\mathcal{O}\cup\mathcal{S}))$. We also suppose that W_ε and $w_{\varepsilon,h}$ are in $L^2_{\text{loc}}(\mathbf{R}^+, \bigcap_{s>0} H^s(\Omega))$, and we can perform the following easy algebraic computations.

We note that $P(w_\varepsilon\cdot\nabla w_\varepsilon) = P(w_\varepsilon\wedge \operatorname{rot} w_\varepsilon)$ so it is enough to compute the weak limit of $\Pi(w_\varepsilon\wedge\operatorname{rot} w_\varepsilon)$. We claim that there is some bounded distribution vector f_ε such that

$$\int w_\varepsilon\wedge\operatorname{rot} w_\varepsilon(x_h,x_3)\,\mathrm{d}x_3 = \varepsilon\partial_t f_\varepsilon(x_h) + \tilde r_\varepsilon - \left(\left(\frac{1}{b^2}\int \rho_{\varepsilon,2}\varepsilon\partial_t\rho_{\varepsilon,1}\,\mathrm{d}x_3\right)\nabla_h b, 0\right), \tag{6}$$

where $\tilde r_\varepsilon$ is a vector field going to zero in $\mathcal{D}'(\mathbf{R}^+\times(\mathcal{O}\cup\mathcal{S}))$ while $\rho_{\varepsilon,h} \stackrel{\text{def}}{=} \nabla_h^\perp W_\varepsilon - w^\perp_{\varepsilon,h}$ and $\rho_{\varepsilon,3}\stackrel{\text{def}}{=}\partial_1 w_{\varepsilon,2} - \partial_2 w_{\varepsilon,1}$, which means that $\operatorname{rot} w_\varepsilon \stackrel{\text{def}}{=} (\partial_3\rho_{\varepsilon,h},\rho_{\varepsilon,3})$. Since we will take the scalar product against a smooth vector field of the type $(\nabla_h^\perp\phi(t,x_h),\alpha(t,x_h))$ with $\nabla_h^\perp\phi\cdot\nabla_h b = 0$, the result will follow directly. So let us prove the claim (6).

We start by recalling that

$$w_{\varepsilon,h} = -\frac{1}{b}(\varepsilon\partial_t\rho_{\varepsilon,h} + r_\varepsilon), \tag{7}$$

where to simplify notation we shall call from now on r_ε any term going strongly to zero in the space $L^2_{\text{loc}}(\mathbf{R}^+\times(\mathcal{O}\cup\mathcal{S}))$, which can change from line to line. An easy computation enables us to infer that

$$w_\varepsilon\wedge\operatorname{rot} w_\varepsilon = \begin{pmatrix} -\frac{1}{b}(\varepsilon\partial_t\rho_{\varepsilon,2}+r_\varepsilon)\rho_{\varepsilon,3} - \partial_3 W_{\varepsilon,3}\partial_3\rho_{\varepsilon,2} \\ \frac{1}{b}(\varepsilon\partial_t\rho_{\varepsilon,1}+r_\varepsilon)\rho_{\varepsilon,3} + \partial_3 W_{\varepsilon,3}\partial_3\rho_{\varepsilon,1} \\ \frac{1}{b}(\varepsilon\partial_t\rho_{\varepsilon,2}+r_\varepsilon)\partial_3\rho_{\varepsilon,1} - \frac{1}{b}(\varepsilon\partial_t\rho_{\varepsilon,1}+r_\varepsilon)\partial_3\rho_{\varepsilon,2}\end{pmatrix}.$$

An integration by parts using the divergence free condition yields

$$\int (w_\varepsilon\wedge\operatorname{rot} w_\varepsilon)_h\,\mathrm{d}x_3 = -\frac{1}{b}\int(\varepsilon\partial_t\rho^\perp_{\varepsilon,h}+r_\varepsilon)\rho_{\varepsilon,3}\,\mathrm{d}x_3 - \int\rho^\perp_{\varepsilon,h}\operatorname{div}_h w_{\varepsilon,h}\,\mathrm{d}x_3.$$

Now we recall that $-\varepsilon\partial_t\rho_{\varepsilon,3} + \operatorname{div}_h(bw_{\varepsilon,h}) = r_\varepsilon$ so it follows that

$$\int (w_\varepsilon\wedge\operatorname{rot} w_\varepsilon)_h\,\mathrm{d}x_3 = -\frac{1}{b}\int\varepsilon\partial_t(\rho^\perp_{\varepsilon,h}\rho_{\varepsilon,3})\,\mathrm{d}x_3 + \frac{1}{b}\int\rho^\perp_{\varepsilon,h}w_{\varepsilon,h}\cdot\nabla_h b\,\mathrm{d}x_3 + \tilde r_\varepsilon(x_h),$$

where $\tilde r_\varepsilon$ denotes now generically a vector going to zero in $\mathcal{D}'(\mathbf{R}^+\times(\mathcal{O}\cup\mathcal{S}))$. So we get

$$\int (w_\varepsilon\wedge\operatorname{rot} w_\varepsilon)_h\,\mathrm{d}x_3 = -\frac{1}{b}\int\varepsilon\partial_t(\rho^\perp_{\varepsilon,h}\rho_{\varepsilon,3})\,\mathrm{d}x_3 - \frac{1}{b^2}\int\rho^\perp_{\varepsilon,h}\varepsilon\partial_t\rho_{\varepsilon,h}\cdot\nabla_h b\,\mathrm{d}x_3 + \tilde r_\varepsilon(x_h),$$

and in particular we have

$$\int (w_\varepsilon\wedge\operatorname{rot} w_\varepsilon)_1\,\mathrm{d}x_3 = -\frac{1}{b}\int\varepsilon\partial_t(\rho_{\varepsilon,2}\rho_{\varepsilon,3})\,\mathrm{d}x_3 - \frac{1}{b^2}\int\rho_{\varepsilon,2}\varepsilon\partial_t\rho_{\varepsilon,1}\partial_1 b\,\mathrm{d}x_3 - \frac{1}{2b^2}\int\varepsilon\partial_t\big(\rho^2_{\varepsilon,2}\partial_2 b\big)\,\mathrm{d}x_3 + \tilde r_\varepsilon(x_h).$$

Similarly

$$\int (w_\varepsilon\wedge\operatorname{rot} w_\varepsilon)_2\,\mathrm{d}x_3 = \frac{1}{b}\int\varepsilon\partial_t(\rho_{\varepsilon,1}\rho_{\varepsilon,3})\,\mathrm{d}x_3 + \frac{1}{b^2}\int\rho_{\varepsilon,1}\varepsilon\partial_t\rho_{\varepsilon,2}\partial_2 b\,\mathrm{d}x_3 + \frac{1}{2b^2}\int\varepsilon\partial_t\big(\rho^2_{\varepsilon,1}\partial_1 b\big)\,\mathrm{d}x_3 + \tilde r_\varepsilon(x_h).$$

But one can also write

$$\int (w_\varepsilon\wedge\operatorname{rot} w_\varepsilon)_2\,\mathrm{d}x_3 = \frac{1}{b}\int\varepsilon\partial_t\left(\rho_{\varepsilon,1}\rho_{\varepsilon,3} + \frac{1}{2b}\rho^2_{\varepsilon,1}\partial_1 b + \frac{1}{b}\rho_{\varepsilon,1}\rho_{\varepsilon,2}\partial_2 b\right)\mathrm{d}x_3 - \frac{1}{b^2}\int\rho_{\varepsilon,2}\varepsilon\partial_t\rho_{\varepsilon,1}\partial_2 b\,\mathrm{d}x_3 + \tilde r_\varepsilon(x_h).$$

It follows that up to full derivatives of the type $\varepsilon\partial_t$ and remainder terms of the type $\tilde{r}_\varepsilon$, the average $\int(w_\varepsilon \wedge \operatorname{rot} w_\varepsilon)_h \, dx_3$ is equal to

$$-\left(\frac{1}{b^2}\int \rho_{\varepsilon,2}\varepsilon\partial_t\rho_{\varepsilon,1}\,dx_3\right)\nabla_h b. \tag{8}$$

So the claim (6) is proved, and Proposition 2.2 follows.

Now to end the proof of the theorem, we still have to find the equation on $\bar{u}$. On $\mathcal{S}$, we identify as usual the 2D1/2 Navier–Stokes equations. On $\mathcal{O}$ the only point left to check is that there is no nonlinear term in the equation. That will imply that $\bar{u}_h$ satisfies a heat-type equation simply because $-(\Pi\Delta_h\bar{u}|\bar{u})_{L^2} = \|\nabla_h\bar{u}\|_{L^2}^2$ since $\Pi\bar{u} = \bar{u}$. So we will prove that

$$\forall\Phi \in \operatorname{Ker} L \cap C_c^\infty(\mathcal{O}), \quad (\bar{u}_h \cdot \nabla_h\bar{u}_h|\Phi_h)_{L^2} = 0. \tag{9}$$

By definition of $\operatorname{Ker}(L)$, we have $\Phi_h \cdot \nabla_h b = \bar{u}_h \cdot \nabla_h b = 0$, from which we infer that

$$\Phi_h \wedge \bar{u}_h = 0. \tag{10}$$

Now we can write $\Pi(\bar{u}_h \cdot \nabla_h\bar{u}_h) = -\Pi(\bar{u}_h \wedge \operatorname{rot}\bar{u}_h)$, hence $(\bar{u}_h \cdot \nabla_h\bar{u}_h|\Phi_h)_{L^2(\mathcal{O})} = (\bar{u}_h \wedge \Phi_h|\operatorname{rot}\bar{u}_h)_{L^2(\mathcal{O})}$. The claim (9) follows, and we refer to [4] for the end of the proof of the theorem.

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