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Differential Topology

Riemann–Poisson manifolds and Kähler–Riemann foliations

Les variétés de Riemann–Poisson et les feuilletages riemanniens de Kähler

Mohamed Boucetta

Faculté des sciences et techniques, BP 549, Marrakech, Morocco

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Abstract

Riemann–Poisson manifolds were introduced by the author in C. R. Acad. Sci. Paris, Ser. I 333 (2001) 763–768, and studied in detail in preprint math.DG/0206102. Kähler–Riemann foliations form an interesting subset of the Riemannian foliations with remarkable properties (see Ann. Global Anal. Geom. 21 (2002) 377–399). In this Note we will show that to give a regular Riemann–Poisson structure on a manifold P is equivalent to give a Kähler–Riemann foliation on P such that the leafwise symplectic form is invariant with respect to all local foliation-preserving perpendicular vector fields. Finally, we give some examples of such manifolds. ***To cite this article: M. Boucetta, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Les structures de Riemann–Poisson ont été introduites par l'auteur dans C. R. Acad. Sci. Paris, Ser. I 333 (2001) 763–768, et étudiées en détail dans Preprint math.DG/0206102. Les feuilletages riemanniens de Kähler forment une partie importante des feuilletages riemanniens et possèdent des propriétés remarquables (voir Ann. Global Anal. Geom. 21 (2002) 377–399). Dans cette Note, nous allons montrer que la donnée d'une structure de Riemann–Poisson sur une variété P équivaut à la donnée d'un feuilletage riemannien de Kähler dont la forme symplecique le long des feuilles est invariante par les champs de vecteurs feuilletés orthogonaux. Nous donnerons aussi des exemples de telles variétés. ***Pour citer cet article : M. Boucetta, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

Soit (P, π) une variété de Poisson. Le tenseur π définit un morphisme fibré $\#_\pi : T^*P \to TP$ par $\beta(\#_\pi(\alpha)) = \pi(\alpha, \beta)$, $\alpha, \beta \in T^*P$, et un crochet de Lie sur l'espace des 1-formes $\Omega^1(M)$ par

$$[\alpha, \beta]_\pi = L_{\#_\pi(\alpha)}\beta - L_{\#_\pi(\beta)}\alpha - d\big(\pi(\alpha, \beta)\big).$$

E-mail address: boucetta@fstg-marrakech.ac.ma (M. Boucetta).

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Soit $\langle,\rangle$ une métrique sur le fibré cotangent T^*P. On considère la connexion contravariante introduite dans [1] et définie par

$$2\langle D^\pi_\alpha \beta, \gamma\rangle = \#_\pi(\alpha)\cdot\langle\beta,\gamma\rangle + \#_\pi(\beta)\cdot\langle\alpha,\gamma\rangle - \#_\pi(\gamma)\cdot\langle\alpha,\beta\rangle + \langle[\alpha,\beta]_\pi,\gamma\rangle + \langle[\gamma,\alpha]_\pi,\beta\rangle + \langle[\gamma,\beta]_\pi,\alpha\rangle.$$

On dira que $(P,\pi,\langle,\rangle)$ est une variété de Riemann–Poisson si $D^\pi\pi = 0$. Cette notion a été introduite dans [1] avec une terminologie différente. Nous obtenons :

Théorème 0.1. *Soit $(P,\pi,\langle,\rangle)$ une variété de Riemann–Poisson régulière. Alors le feuilletage symplectique est un feuilletage riemannien de Kähler. Inversement, étant donné une variété $(P,\mathcal{F},g)$ munie d'un feuilletage riemannien $\mathcal{F}$ et d'une métrique quasi-fibrée g. On suppose qu'il existe $\omega \in \Gamma(\Lambda^2 T^*\mathcal{F})$ telle que :*

(1) *pour chaque feuille S, la restriction de ω à S est une forme symplectique parallèle pour la connexion de Levi-Civita associée à la restriction de g à S ;*
(2) *pour tout champ feuilleté local X ortohogonal au feuilletage et pour tout couple (U,V) de champs de vecteurs locaux et tangents au feuilletage, $L_X\omega(U,V) = 0$.*

Alors il existe un tenseur de Poisson π sur P et une métrique $\langle,\rangle$ sur T^*P tels que $(P,\pi,\langle,\rangle)$ soit une variété de Riemann–Poisson régulière dont le fuilletage symplectique est $\mathcal{F}$.

Théorème 0.2. *Soit G un goupe de Lie connexe et $\mathcal{G}$ son algèbre de Lie. On suppose qu'il existe une métrique bi-invariante $\langle,\rangle$ sur T^*G. Soit $r \in \mathcal{G}\wedge\mathcal{G}$ et π^r le champ de bi-vecteur obtenu à partir de r par translations à gauche. $(G,\pi^r,\langle,\rangle)$ est une variété Riemann–Poisson si et seulement si*

$$r(ad^*_{r(\alpha)}\beta,\gamma) + r(\beta, ad^*_{r(\alpha)}\gamma) = 0 \quad \forall \alpha,\beta,\gamma \in \mathcal{G}^*,$$

où r désigne aussi l'application linéaire de $\mathcal{G}^*$ vers $\mathcal{G}$ définie par r.

1. Statement of results

Many fundamental definitions and results about Poisson manifolds can be found in Vaisman's monograph [5].

Let P be a Poisson manifold with Poisson tensor π. We have a bundle map $\#_\pi : T^*P \to TP$ defined by

$$\beta(\#_\pi(\alpha)) = \pi(\alpha,\beta), \quad \alpha,\beta \in T^*P. \tag{1}$$

On the space of differential 1-forms $\Omega^1(P)$, the Poisson tensor induces a Lie bracket

$$[\alpha,\beta]_\pi = L_{\#_\pi(\alpha)}\beta - L_{\#_\pi(\beta)}\alpha - d(\pi(\alpha,\beta)). \tag{2}$$

For this Lie bracket and the usual Lie bracket on vector fields, the bundle map $\#_\pi$ induces a Lie algebra homomorphism $\#_\pi : \Omega^1(P) \to \mathcal{X}(P)$:

$$\#_\pi([\alpha,\beta]_\pi) = [\#_\pi(\alpha), \#_\pi(\beta)]. \tag{3}$$

Let $\langle,\rangle$ be a metric on the cotangent bundle T^*P. We consider the contravariant connection D^π associated with the couple $(\pi,\langle,\rangle)$ and defined in [1] by

$$\begin{aligned}2\langle D^\pi_\alpha\beta,\gamma\rangle = {}& \#_\pi(\alpha)\cdot\langle\beta,\gamma\rangle + \#_\pi(\beta)\cdot\langle\alpha,\gamma\rangle - \#_\pi(\gamma)\cdot\langle\alpha,\beta\rangle \\ & + \langle[\alpha,\beta]_\pi,\gamma\rangle + \langle[\gamma,\alpha]_\pi,\beta\rangle + \langle[\gamma,\beta]_\pi,\alpha\rangle,\end{aligned} \tag{4}$$

where $\alpha,\beta,\gamma \in \Omega^1(P)$.

D^π satisfies:

$$D^{\pi}_{\alpha}\beta - D^{\pi}_{\beta}\alpha = [\alpha,\beta]_{\pi}; \tag{5}$$

$$\#_{\pi}(\alpha)\cdot\langle\beta,\gamma\rangle = \langle D^{\pi}_{\alpha}\beta,\gamma\rangle + \langle\beta, D^{\pi}_{\alpha}\gamma\rangle. \tag{6}$$

The following definition was given in [1] with a different terminology.

Definition 1.1. With the notations above, the triple $(P,\pi,\langle,\rangle)$ is called a Riemann–Poisson manifold if $D^{\pi}\pi = 0$.

A regular Riemann–Poisson manifold is a Riemann–Poisson manifold with regular symplectic foliation.

Let $(P,\pi,\langle,\rangle)$ be a regular Riemann–Poisson manifold. We denote $\mathcal{F}$ the associated symplectic foliation and $T\mathcal{F}$ the involutive distribution tangent to this foliation. The leafwise symplectic form ω belongs to $\Gamma(\Lambda^2 T^*\mathcal{F})$ and is given by

$$\omega(u,v) = \pi\big(\pi^{-1}(u),\pi^{-1}(v)\big), \quad u,v\in T\mathcal{F}, \tag{7}$$

where $\pi^{-1}(u)$ and $\pi^{-1}(v)$ denote any antecedents by $\#_{\pi}$ of u and v.

We have

$$T^*P = \operatorname{Ker}\pi \oplus (\operatorname{Ker}\pi)^{\perp}, \tag{8}$$

where $(\operatorname{Ker}\pi)^{\perp}$is the orthogonal of $(\operatorname{Ker}\pi)$ with respect to $\langle,\rangle$. The metric gives an identification between the cotangent bundle T^*P and the tangent bundle TP which we denote $\# : T^*P \to TP$. We put $\mathcal{H} = \#(\operatorname{Ker}\pi)$. We get

$$TP = T\mathcal{F}\oplus\mathcal{H} \quad \text{and} \quad \#\big((\operatorname{Ker}\pi)^{\perp}\big) = T\mathcal{F}. \tag{9}$$

The bundle map $\#_{\pi}$ induces an isomorphism $\#_{\pi} : (\operatorname{Ker}\pi)^{\perp} \to T\mathcal{F}$; we denote ${\#_{\pi}}^{-1} : T\mathcal{F} \to (\operatorname{Ker}\pi)^{\perp}$ its inverse.

We define a Riemannian metric on TP by

$$\begin{aligned} g_{\pi}\big(\#(\alpha),\#(\beta)\big) &= \langle\alpha,\beta\rangle, \quad \alpha,\beta\in\operatorname{Ker}\pi;\\ g_{\pi}(u,v) &= \big\langle {\#_{\pi}}^{-1}(u), {\#_{\pi}}^{-1}(v)\big\rangle, \quad u,v\in T\mathcal{F};\\ g_{\pi}\big(u,\#(\alpha)\big) &= 0, \quad \alpha\in\operatorname{Ker}\pi, u\in T\mathcal{F}. \end{aligned}$$

We recall the definition of a Kähler–Riemann foliation (see [3]). A foliation $\mathcal{F}$ on a manifold is called a Kähler foliation if it is endowed with a complex structure J and Hermitian metric $h = S - 2\mathrm{i}\omega$ on $T\mathcal{F}$ such that $d_{\mathcal{F}}\omega = 0$. A Kähler foliation which is also a Riemannian foliation is called Kähler–Riemann foliation.

We recall also the definition of a foliation-preserving vector field. Let $(P,\mathcal{F})$ be a foliated manifold. A vector field $X\in\mathcal{X}(P)$ is said to be foliation-preserving if, for all Y tangent to $T\mathcal{F}$, $[X,Y]$ is tangent to $T\mathcal{F}$.

Theorem 1.1. *Let $(P,\pi,\langle,\rangle)$ be a regular Riemann–Poisson manifold. Then the symplectic foliation is a Kähler–Riemann foliation and g_{π} is a bundle-like metric.*

Conversely, let $(P,\mathcal{F},g)$ be a differentiable manifold endowed with a Riemannian foliation $\mathcal{F}$ and a bundle-like metric g. We suppose that there is $\omega\in\Gamma(\Lambda^2T^*\mathcal{F})$ such that:

(1) *for any leaf S, the restriction of ω to S is symplectic and parallel with respect to the Levi-Civita connection associated with the restriction of g to S*;
(2) *for any local perpendicular foliation-preserving vector field X and any couple (U,V) of local vector fields tangent to F,*

$$L_X\omega(U,V) = 0.$$

Then there is a Poisson tensor π on P and metric $\langle,\rangle$ on T^*P such that $(P,\pi,\langle,\rangle)$ is a regular Riemann–Poisson manifold whose symplectic foliation is $\mathcal{F}$.

Theorem 1.2. *Let G be a connected Lie group with the Lie algebra $\mathcal{G}$. We suppose that there is a bi-invariant metric $\langle,\rangle$ on T^*G. Let $r\in\mathcal{G}\wedge\mathcal{G}$ and denote π^r the bi-vector field defined from r by left translations. $(G,\pi^r,\langle,\rangle)$ is a Riemann–Poisson manifold if and only if*

$$r(ad^*_{r(\alpha)}\beta,\gamma)+r(\beta,ad^*_{r(\alpha)}\gamma)=0\quad\forall\alpha,\beta,\gamma\in\mathcal{G}^*\tag{10}$$

and where r denotes also the linear map from $\mathcal{G}^*$ to $\mathcal{G}$ induced by r.

2. Sketch of the proof of Theorem 1.1

2.1. Preliminaries

The notations and definitions are those given in Section 1.

The following propositions give some elementary properties of a regular Riemann–Poisson manifold.

Proposition 2.1. *Let $(P,\pi,\langle,\rangle)$ be a regular Riemann–Poisson manifold. Let D^π be the contravariant connection associated with $(\pi,\langle,\rangle)$. Then*

(1) $\#_\pi(\beta)=0\Rightarrow\forall\alpha\in\Omega^1(P),\ \#_\pi(D^\pi_\alpha\beta)=0$;
(2) $\#_\pi(\alpha)=0\Rightarrow D^\pi_\alpha=0$;
(3) *If* $\alpha,\beta\in\Gamma((\operatorname{Ker}\pi)^\perp)$ *then* $D^\pi_\alpha\beta\in\Gamma((\operatorname{Ker}\pi)^\perp)$ *and* $[\alpha,\beta]_\pi\in\Gamma((\operatorname{Ker}\pi)^\perp)$.

Proposition 2.2. *Let $(P,\pi,\langle,\rangle)$ be a regular Riemann–Poisson manifold. We have*

$$L_X\pi(\alpha,\beta)=0,\tag{11}$$

for any $\alpha,\beta\in\Gamma((\operatorname{Ker}\pi)^\perp)$ and for any X tangent to $\mathcal{H}$.

Now, let $(P,\pi,\langle,\rangle)$ be a regular Riemann–Poisson manifold. Let $\mathcal{H}$ be the orthogonal distribution to the symplectic foliation $\mathcal{F}$ defined by (9). We will gives an interesting characterization of the foliation-preserving vector fields which are tangent to $\mathcal{H}$.

We denote $\Omega^0_b(P)$ the space of Casimir functions, i.e., those functions which are constant on the symplectic leaves and $\Omega^1_b(P)$ the space of basic differential 1-forms. An 1-form α belongs to $\Omega^1_b(P)$ if and only if

$$\#_\pi(\alpha)=0\quad\text{and}\quad i_{\#_\pi(\beta)}d\alpha=0,\quad\forall\beta\in\Omega^1(P).$$

We have from (2) and a careful verification that

$$\alpha\in\Omega^1_b(P)\quad\Longleftrightarrow\quad\forall\beta\in\Omega^1(P),\quad[\alpha,\beta]_\pi=0.\tag{12}$$

Proposition 2.3. *Let $(P,\pi,\langle,\rangle)$ be a regular Riemann–Poisson manifold. If $\alpha\in\Gamma(\operatorname{Ker}\pi)$, the following assertions are equivalent.*

(1) α *is a basic 1-form.*
(2) $D^\pi\alpha=0$.
(3) $\#(\alpha)$ *is a foliation-preserving vector field.*
(4) $L_{\#(\alpha)}\pi=0$.

Furthermore, if $\alpha,\beta\in\Omega^1_b(P)$ then $\langle\alpha,\beta\rangle$ is a Casimir function.

2.2. Sketch of the proof of Theorem 1.1

Let $(P, \pi, \langle,\rangle)$ be a regular Riemann–Poisson manifold. Let us remind that the notations and definitions are those of Section 1.

Let S be a symplectic leave. We denote g_S and ω_S the restrictions of g_π and ω to S. The Levi-Civita connection ∇^S of g_S is given by (see [1])

$$\nabla^S_{\#_\pi(\alpha)}\#_\pi(\beta) = \#_\pi\big(D^\pi_\alpha\beta\big), \quad \alpha, \beta \in \Gamma\big((\operatorname{Ker}\pi)^\perp\big) \tag{13}$$

and thus we have

$$\nabla^S\omega_S = 0. \tag{14}$$

Thus, S is a Kähler manifold (see [2]) and then the symplectic foliation is a Kähler foliation.

We show now that g_π is bundle-like. Following Reinhart [4], the metric g_π is said to be bundle-like if it has the following property: for any open set U in P and for all vector fields $Y; Z$ that are foliation-preserving and perpendicular to the leaves, the function $g_\pi(Y, Z)$ is a basic function on U.

In our case, the perpendicular foliation-preserving vector fields are $Y = \#(\alpha)$ and $Z = \#(\beta)$ where α, β are basic 1-forms. Furthermore $g_\pi(Y, Z) = \langle\alpha, \beta\rangle$ which is a Casimir function by Proposition 2.3 and then g_π is a bundle-like metric.

Now we give the converse.

We have

$$TP = T\mathcal{F} \oplus T^\perp\mathcal{F}, \qquad T^*P = T^\circ\mathcal{F} \oplus T^{\perp\circ}\mathcal{F},$$

where $T^\perp\mathcal{F}$ is the distribution g-orthogonal to $T\mathcal{F}$ and $T^\circ\mathcal{F} = \{\alpha \in T^*P \mid \alpha(T\mathcal{F}) = 0\}$.

Denote $\# : T^*P \to TP$ the identification given by the Riemannian metric g. We have

$$\#(T^\circ\mathcal{F}) = T^\perp\mathcal{F} \quad \text{and} \quad \#\big(T^{\perp\circ}\mathcal{F}\big) = T\mathcal{F}.$$

The leafwise symplectic form ω can be considered as a 2-form on P by setting $i_v\omega = 0$ for any $v \in T^\perp\mathcal{F}$ and hence realizes un isomorphism $\omega : T\mathcal{F} \to T^{\perp\circ}\mathcal{F}$, $v \mapsto \omega(v, \cdot)$. Denote $\omega^{-1} : T^{\perp\circ}\mathcal{F} \to T\mathcal{F}$ its inverse.

Now, define a bi-vector π by

$$\pi(\alpha, \beta) = \begin{cases} \omega\big(\omega^{-1}(\alpha), \omega^{-1}(\beta)\big) & \text{if } \alpha, \beta \in T^{\perp\circ}\mathcal{F}, \\ 0 & \text{if } \alpha \text{ or } \beta \text{ belongs to } T^\circ\mathcal{F}, \end{cases}$$

and a metric on T^*P by

$$\langle\alpha, \beta\rangle = \begin{cases} g\big(\omega^{-1}(\alpha), \omega^{-1}(\beta)\big) & \text{if } \alpha, \beta \in T^{\perp\circ}\mathcal{F}, \\ g\big(\#(\alpha), \#(\beta)\big) & \text{if } \alpha, \beta \in T^\circ\mathcal{F}, \\ 0 & \text{if } \alpha \in T^{\perp\circ}\mathcal{F} \text{ and } \beta \in T^\circ\mathcal{F}. \end{cases}$$

Now, it is a straightforward calculation to compute the contravariant connection D^π associated with $(\pi, \langle,\rangle)$ and to show that $D^\pi\pi = 0$. □

The condition (10) is satisfied if r is bi-invariant. The bi-invariant Poisson structures on a Lie group were characterized by Vaisman in [5]. The condition (10) is also satisfied if $\operatorname{Im} r$ lies in the center of $\mathcal{G}$. Then, for any symplectic subspace (S, ω) in the center of G, i.e., a vector subspace S with a non-degenerate bilinear from ω, the bi-vector r associated to the linear map $\mathcal{G} \xrightarrow{i^*} S^* \xrightarrow{\omega^{-1}} S$ satisfies (10) which gives a method to construct many examples of Riemann–Poisson manifolds. We can get more complicated examples by considering a discrete subgroup Γ of a Riemann–Poisson manifold G as in the theorem above. On G/Γ there is a Riemann–Poisson structure.

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