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COMPTES RENDUS MATHEMATIQUE

Harmonic Analysis/Mathematical Analysis

A null series with small anti-analytic part

Une série trigonométrique qui converge presque partout vers zero et dont la partie antianalytique est de carré intégrable

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Abstract

We show that it is possible for an L^2 function on the circle, which is a sum of an almost everywhere convergent series of exponentials with positive frequencies, to not belong to the Hardy space H^2. A consequence in the uniqueness theory is obtained. ***To cite this article: G. Kozma, A. Olevskiĭ, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Il existe une série trigonométrique dont toutes les fréquences sont positives et qui converge presque partout vers une fonction de carré intégrable qui admet des fréquences négatives. Ce fait est équivalent à l'existence de la série trigonométrique mentionnée dans le titre. Il s'agit donc d'une contribution à la théorie de l'unicité du développement trigonométrique. ***Pour citer cet article : G. Kozma, A. Olevskiĭ, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

1. Introduction

A (nontrivial) trigonometric series

$$\sum c(n)\, \mathrm{e}^{\mathrm{i}nt}, \quad t \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}, \tag{1}$$

is called a null series if it converges to zero almost everywhere (a.e.). The existence of such a series was discovered by Menshov in 1916 (see [1, Chapter XIV]). He constructed a singular compactly supported finite Borel measure on $\mathbb{T}$ with Fourier transform vanishing at infinity. The Riemannian theory implies that the Fourier series of this measure converges to zero at every point outside of the support. This famous example of Menshov was the origin of the modern uniqueness theory in Fourier Analysis, see [1,4,5].

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Clearly a null series cannot belong to L^2. A less trivial observation is that it cannot be "analytic" that is involve positive frequencies only. This follows from the Abel summation and Privalov "angular limit" theorems. It turns out however that the "non-analytic" part of a null series may belong to L^2.

Theorem 1.1. *There is a null series* (1) *such that* $\sum_{n<0} |c(n)|^2 < \infty$.

An equivalent formulation of the result:

Theorem 1.2. *There is a power series* $F(z) = \sum c(n)z^n$ *converging a.e. on the circle* $|z| = 1$ *to some function* $f \in L^2(\mathbb{T})$, *and* f *does not belong to the Hardy space* H^2.

When we say that a function f on the circle is in H^2 we mean that it is a boundary limit of an H^2 function on the disk, or, equivalently, that $f \in L^2$ and $\hat{f}(-n) = 0$ for $n = 1, 2, \ldots$.

To see that Theorem 1.1 implies 1.2, use Carleson's convergence theorem [3] to get that the analytic part of the sum, $\sum_{n\geqslant 0} c(n)z^n$ converges a.e. on the circle $|z| = 1$ and then use, as above, Abel and Privalov's theorems to get that the resulting f is not in H^2. Reversing these arguments one may derive Theorem 1.1 from Theorem 1.2.

It should be mentioned that usually if representation by harmonics is unique then it is the Fourier series. Compare for instance the classical Cantor and du Bois–Reymond [1, pp. 193, 201] theorems on pointwise convergence everywhere. Our result shows that this principle is not universal. Indeed, any f may have *at most one representation* by an a.e. convergent series

$$\sum_{n\geqslant 0} c(n)\,\mathrm{e}^{\mathrm{i}nt} \tag{2}$$

however, even if $f \in L^2$, the coefficients, in general, cannot be recovered by Fourier's formula.

2. Proof

2.1. Our main goal is to construct an "analytic pseudofunction", that is

$$F(z) = \sum_{n\geqslant 0} c(n)z^n, \tag{3}$$

$$c(n) = \mathrm{o}(1) \tag{4}$$

with the following properties:

(i) $F \notin H^2$.
(ii) There is a compact $K \subset \mathbb{T}$ of Lebesgue measure zero such that F has boundary values on K^c $f(t) := \lim_{z\to \mathrm{e}^{\mathrm{i}t}} F(z)\ \forall t \notin K$.
(iii) $f \in L^\infty(\mathbb{T})$.
(iv) The limit is uniform on any closed arc $J \subset K^c$.

Having a function F with all the properties above one can get the result easily. Indeed, the series (3) on the boundary represents a distribution $\overline{F} := \sum c(n)\,\mathrm{e}^{\mathrm{i}nt}$, which is the limit (in distributional sense) of $F_r := F(r\,\mathrm{e}^{\mathrm{i}t})$ as $r \to 1$. On the other hand $F(r\,\mathrm{e}^{\mathrm{i}t}) \to f(t)$ uniformly on any closed arc $J \subset K^c$, so the distribution $\overline{F} - f$ is supported on K. Hence the condition (4) implies uniform convergence of the Fourier series of $\overline{F} - f$ to zero on any such J, see [4, p. 54]. Theorem 1.2 (and hence Theorem 1.1) will follow.

The function F will be obtained as $1/G$ where G is a singular inner function, so

$$F(z) = \exp\biggl(\int_{\mathbb{T}} \frac{\mathrm{e}^{\mathrm{i}t} + z}{\mathrm{e}^{\mathrm{i}t} - z}\,\mathrm{d}\mu(t)\biggr). \tag{5}$$

This construction will ensure (i)–(iv), if μ is a positive measure supported on K, so our task in Sections 2.2–2.4 will be to construct a singular μ such that (4) will be satisfied.

2.2. Denoting $g(x) := x\,\mathrm{e}^{2/x} + 1 - x$ we fix a sequence $l(1) > l(2) > \cdots \to 0$, such that $g(l(n)) - g(l(n-1)) = \mathrm{o}(1)$.

Proceed with the induction as follows. Let $K_0 = \mathbb{T}$. Suppose we already have a compact $K_{n-1} \subset \mathbb{T}$ which is a finite union of segments of equal lengths. Divide each of them to $q(n)$ equal subsegments I and replace each I by the concentric segment I',

$$|I'| = \frac{l(n)}{l(n-1)}|I|$$

(here and below by $|E|$ we denote the normalized Lebesgue measure of a set $E \subset \mathbb{T}$). Set $K_n := \bigcup I'$, so $|K_n| = l(n)$, and

$$u_n := \frac{1}{l(n)}\mathbf{1}_{K_n}.$$

Claim 1. *If the number $q = q(n)$ is sufficiently large then the function $u_n - u_{n-1}$ is "almost orthogonal" to any pre-given finite dimensional subspace in $L^2(\mathbb{T})$. More precisely: for any $\varepsilon > 0$, $N \in \mathbb{N}$ there is a $Q \in \mathbb{N}$ such that $\forall q(n) > Q$, $|\widehat{u_n - u_{n-1}}(k)| < \varepsilon$ $\forall k$, $|k| < N$ (here and below the sign $\hat{\ }$ stands for the Fourier transform on $\mathbb{T}$. The term "sufficiently large" means that the minimal allowed value may depend on everything that happened in previous stages of the induction).*

To prove the claim it is enough to mention that $u_n - u_{n-1}$ is supported on the union of the segments I, the length of each I is arbitrary small as q gets large and the average of $u_n - u_{n-1}$ on I equals to zero.

Obviously the same inequality holds for the conjugate function $\widetilde{u_n - u_{n-1}}$ so we obtain

Claim 2. *Given $\varepsilon > 0$, $N \in \mathbb{N}$ and sufficiently large q the function $h_n := (u_n - u_{n-1}) + \mathrm{i}(\widetilde{u_n - u_{n-1}})$ satisfies $|\widehat{h_n}(k)| < \varepsilon$ $\forall k$, $0 \leqslant k < N$.*

Now denote: $f_n = \mathrm{e}^{u_n + \mathrm{i}\widetilde{u_n}}$.

Claim 3. *Given $\varepsilon > 0$, $N \in \mathbb{N}$ and q sufficiently large we have: $|\widehat{f_n - f_{n-1}}(k)| < \varepsilon$ $\forall k$, $0 \leqslant k < N$.*

Indeed, $f_n - f_{n-1} = f_{n-1}(\mathrm{e}^{h_n} - 1)$.

Clearly the fact that h_n are analytic gives that they may be exponentiated formally (e.g., by extending to the disk $\mathbb{D}$ and using $\hat{h}(k) = h^{(k)}(0)$) which gives that $\widehat{\mathrm{e}^{h_n} - 1}(k)$ is a polynomial with no constant term in $\widehat{h_n}(1), \ldots, \widehat{h_n}(k)$. Since f_{n-1} is also analytic, $\widehat{f_n - f_{n-1}}(k)$ is a finite combination of $\widehat{f_{n-1}}(j)$ and $\widehat{\mathrm{e}^{h_n} - 1}(k - j)$ and the claim is a consequence of Claim 2.

As an immediate corollary we obtain:

Claim 4. *For any $\varepsilon > 0$ and sufficiently large q $|\langle f_n - f_{n-1}, f_{n-1}\rangle| < \varepsilon$.*

We mean here the usual inner product, $\langle f, g\rangle = \int_{\mathbb{T}} f\bar{g}$.

2.3. Proceeding with the induction above we get

Claim 5. *If the numbers $q(n)$ grow sufficiently fast then there are numbers $N_1 < N_2 < \cdots$ such that the functions $\{f_n\}$ satisfy, for any n, the conditions:*

(i) $|\widehat{f_{n-1}}(k)| < \frac{1}{n}$, for all k such that $k > N_n$.

(ii) $|\widehat{f_n - f_{n-1}}(k)| < 2^{-n}$, for all k such that $0 \leqslant k \leqslant N_n$.
(iii) $|\langle f_n - f_{n-1}, f_n\rangle| < \frac{1}{n}$.

It is enough on the nth step of the induction to choose N_n so that (i) is fulfilled and then to use Claims 3 and 4 to ensure (ii) and (iii).

Let the sequence $\{q(n)\}$ above be fixed. The "almost orthogonality" condition (iii) implies the "almost Pythagorean" equality: $\|f_{n-1}\|^2 + \|f_n - f_{n-1}\|^2 = \|f_n\|^2 + \mathrm{o}(1)$.

From Section 2.2 we have $\|f_n\|^2 = g(l(n))$, so $\|f_n - f_{n-1}\|^2 = \mathrm{o}(1)$. Together with (i) and (ii) this easily gives

$$\|\widehat{f_n - f_m}\|_\infty \to 0 \quad \text{as } n, m \to \infty \tag{6}$$

2.4. Let μ be the weak limit of the measures $\mu_n(\mathrm{d}t) := u_n(t)\,\mathrm{d}t$. Clearly it is a positive measure supported on $K := \bigcap K_n$ and $|K| = 0$. Define F by (5) and F_n by the same formula with μ replaced by μ_n. Then $F_n \to F$ uniformly on compacts inside the unit disc $\mathbb{D}$. Therefore each coefficient $c(k)$ of the expansion (3) may be obtained as the limit of the corresponding coefficients $c_n(k)$ which are just $\widehat{f_n}(k)$, so (6) implies (4) which finishes the proof.

Remark 1. It should be noted that our use of Carleson's convergence theorem to prove the equivalence of Theorems 1.1 and 1.2 is unnecessary, since Theorem 1.1 may be proved directly using the fact (which is easy to see) that the function f defined in Section 2.1 is smooth on any closed arc $J \subset K^c$.

3. Remarks

3.1. In contrast to Menshov's original example, the series (1) in Theorem 1.1 cannot be the Fourier series of a measure. Indeed, if $\mu \sim \sum c(n)\,\mathrm{e}^{\mathrm{i}nt}$ and $\sum_{n<0} |c(n)|^2 < \infty$, them μ must be absolutely continuous, and cannot generate a null-series. See, for example, [1, Section VIII.12].

3.2. Another contrast with the "non-analytic" situation appears when one considers the size of the exceptional set. It is well known that a null series (1) may converge to zero outside a "thin" compact (of zero Hausdorff dimension). On the other hand the following proposition is true

Theorem 3.1. *If a series* (2) *converges to* $f \in L^1(\mathbb{T})$ *everywhere on* $\mathbb{T}$ *outside some set of dimension* < 1 *then it is the Fourier series of* f.

This follows from a Phragmén–Lindelöf type theorem for analytic functions in $\mathbb{D}$ of slow growth, see [2, Theorem 5].

3.3. Let $\mathcal{P}$ be the class of functions in L^2 which can be represented by an a.e. converging sum (2). Theorem 1.2 shows us that $\mathcal{P} \setminus H^2$ is non-trivial. Further, the proof actually gives a little more: $(\mathcal{P} \setminus H^2) \cap L^\infty \neq \emptyset$. The class $\mathcal{P}$ has some interesting properties. We plan to analyze it in a separate paper.

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