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Mathematical Physics

On sum rules of special form for Jacobi matrices

Sur des règles de sommation pour des matrices de Jacobi

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Abstract

We use sum rules of a special form to study spectral properties of Jacobi matrices. As a consequence of the main theorem, we obtain a discrete counterpart of a result by Molchanov, Novitskii and Vainberg (Comm. Math. Phys. 216 (2001) 195–213). ***To cite this article: S. Kupin, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Nous appliquons les règles de sommation de Case à l'étude de propriétés spectrales de matrices de Jacobi d'un certain type. Nous obtenons un analogue discret d'un résultat de Molchanov, Novitskii and Vainberg (Comm. Math. Phys. 216 (2001) 195–213) comme un des corollaires du théorème principal. ***Pour citer cet article : S. Kupin, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

1. Introduction

Recently, the Case sum rules [1,2] were efficiently used to relate properties of elements of a Jacobi matrix of certain class with its spectral properties and vice versa. For instance, spectral data of Jacobi matrices being a Hilbert–Schmidt perturbation of the free Jacobi matrix J_0 (see (1)) were characterized in [4]. Different classes of Jacobi matrices were studied in [5,6]. However, the sum rules become more and more complex with increasing order. In this note, we suggest a modification of the method that permits us to work with higher order sum rules. In particular, we obtain sufficient conditions for a Jacobi matrix to satisfy certain constraints on its spectral measure (see Theorem 1.1).

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Let

$$J = J(a,b) = \begin{bmatrix} b_0 & a_0 & 0 \\ a_0 & b_1 & \ddots \\ 0 & \ddots & \ddots \end{bmatrix}, \qquad J_0 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & \ddots \\ 0 & \ddots & \ddots \end{bmatrix} \tag{1}$$

be Jacobi matrices and $a = \{a_k\}$, $a_k > 0$, $b = \{b_k\}$, $b_k \in \mathbb{R}$. A scalar spectral measure $\sigma = \sigma(J)$ of the matrix J is defined by the relation

$$\big((J - z)e_0, e_0\big) = \int\limits_{\mathbb{R}} \frac{\mathrm{d}\sigma(x)}{x - z}$$

with $z \in \mathbb{C} \setminus \mathbb{R}$. Assuming J to be a compact perturbation J_0, we have that the absolutely continuous spectrum $\sigma_{ac}(J)$ of J fills in $[-2, 2]$, and the discrete spectrum consists of two sequences $\{x_j^{\pm}\}$ with properties $x_j^- < -2$, $x_j^- \to -2$, and $x_j^+ > 2$, $x_j^+ \to 2$.

Let $\partial a = \{a_k - a_{k-1}\}$. For a given a and a $k \in \mathbb{N}$, we construct a sequence $\gamma_k(a)$ by formula

$$\big(\gamma_k(a)\big)_j = \alpha_j^k - \alpha_j \cdots \alpha_{j+k-1},$$

where $\alpha = a - 1$ and 1 is a sequence of units.

Theorem 1.1. *Let $J = J(a,b)$ be a Jacobi matrix described above. If*

$$\text{(i)} \quad a - 1, b \in l^{m+1}, \quad \partial a, \partial b \in l^2, \qquad \text{(ii)} \quad \gamma_k(a) \in l^1, \quad k = 3, [m/2 + 1], \tag{2}$$

then

$$\text{(i}'\text{)} \quad \int\limits_{-2}^{2} \log \sigma'(x) \cdot \big(4 - x^2\big)^{m-1/2} \,\mathrm{d}x > -\infty, \qquad \text{(ii}'\text{)} \quad \sum_j \big(x_j^{\pm 2} - 4\big)^{m+1/2} < \infty. \tag{3}$$

When $m = 1$, the theorem gives a half of [4], Theorem 1.

Of course, relation (2)(ii) is true in the case of a discrete Schrödinger operator, i.e., when $J = J(1, b)$.

Corollary 1.2. *Let $J = J(1,b)$. If $b \in l^{m+1}$, $\partial b \in l^2$, then inequalities* (3) *hold.*

Note also that assumptions of Theorem 1.1 may be slightly weakened in this setting. Namely, the corollary is still true if $b \in l^{m+2}$, m being even. The corollary is a direct counterpart of a result from [7] for a "continuous" Schrödinger operator on a half-line.

2. Proof of Theorem 1.1

The main tool used in the proof is a sum rule of a special type, see [4,6,8,9] in this connection. First, we obtain it assuming $\operatorname{rank}(J - J_0) < \infty$. The passage to the limit is carried out later.

Applying methods of [9], we see that

$$\frac{1}{2\pi} \int\limits_{-2}^{2} \log \frac{\sqrt{4 - x^2}}{\sigma'(x)} \cdot \big(4 - x^2\big)^{m-1/2} \,\mathrm{d}x + \sum_j G_m\big(x_j^{\pm}\big) = \Psi_m(J),$$

where $\Psi_m(J) = \Psi_m(a,b)$, and $G_m(x) = (-1)^{m+1} C_0 (x^2 - 4)^{m+1/2} + \mathrm{O}((x^2 - 4)^{m+3/2})$ with $x \in \mathbb{R} \setminus [-2, 2]$, C_0 being a positive constant. An elementary, but long and tedious computation gives that

$$\Psi_m(J) = \operatorname{tr} \left\{ \sum_{k=1}^{m} \frac{(-1)^{k+1}}{2^{2k+1} k} \widetilde{C}_{2m-1}^{2k-1} \big(J^{2k} - J_0^{2k}\big) - \frac{(2m-1)!!}{(2m)!!} \log A \right\}, \tag{4}$$

where $A = \operatorname{diag}\{a_k\}$ and $\widetilde{C}_m^k = \frac{m!!}{(m-k)!!k!!}$. Notation $k!!$ is used for "even" or "odd" factorials.

The following lemma plays a central role in the whole proof.

Lemma 2.1. *Let $J = J(a,b)$. We have*

$$\left|\Psi_m(J)\right| \leqslant C_1\left(\|a-1\|_{m+1}^{m+1} + \|b\|_{m+1}^{m+1} + \|\partial a\|_2^2 + \|\partial b\|_2^2 + \sum_{k=3}^{[m/2+1]} \left\|\gamma_k(a)\right\|_1\right), \tag{5}$$

where C_1 depends on $\|a-1\|_\infty$, $\|b\|_\infty$, $\|\partial a\|_\infty$, and $\|\partial b\|_\infty$.

Above, norms $\|\cdot\|_p$ refer to the standard l^p-space norms. With exception of the lemma, the proof of Theorem 1.1 goes along standard lines (see [4–6,8]). We quote only its main steps.

Proof of Theorem 1.1. Define $\Phi_m(J)$ as

$$\Phi_m(J) = \Phi_m(\sigma) = \Phi_{m,1}(\sigma) + \Phi_{m,2}(\sigma) = \frac{1}{2\pi}\int_{-2}^{2} \log\frac{\sqrt{4-x^2}}{\sigma'(x)} \cdot \left(4-x^2\right)^{m-1/2}\mathrm{d}x + \sum_j G_m\left(x_j^{\pm}\right).$$

We have to show that $\Phi_m(J) < \infty$. We put $a_N = \{(a_N)_k\}$ and $a'_N = \{(a'_N)_k\}$, where

$$(a_N)_k = \begin{cases} a_k, & k \leqslant N, \\ 1, & k > N, \end{cases} \qquad (a'_N)_k = \begin{cases} 1, & k \leqslant N, \\ a_k, & k > N. \end{cases}$$

Define sequences b_N, b'_N in the same way (of course, with 1's replaced by 0's). Let $J_N = J(a_N, b_N)$ and σ_N be its spectral measure. As we readily see, $a'_N - 1, b'_N \to 0$, $\partial a'_N, \partial b'_N \to 0$, and $\gamma_k(a'_N) \to 0$ in corresponding norms, as $N \to \infty$. By Lemma 2.1, we have for $N' = N - m$

$$\begin{aligned}\left|\Psi_m(J) - \Psi_m(J_N)\right| &\leqslant \left|\Psi_m(a'_{N'}, b'_{N'})\right| \\ &\leqslant C_1\left(\left\|a'_{N'} - 1\right\|_{m+1}^{m+1} + \left\|b'_{N'}\right\|_{m+1}^{m+1} + \left\|\partial a'_{N'}\right\|_2^2 + \left\|\partial b'_{N'}\right\|_2^2 + \sum_k \left\|\gamma_k(a'_{N'})\right\|_1\right),\end{aligned}$$

or, $\Psi_m(J_N) \to \Psi_m(J)$, as $N \to \infty$. On the other hand, $(J_N - z)^{-1} \to (J - z)^{-1}$, for $z \in \mathbb{C} \setminus \mathbb{R}$, and, consequently, $\sigma_N \to \sigma$ weakly. Looking at [4], Corollary 5.3 and Theorem 6.2, we get $\Phi_{m,1}(\sigma) \leqslant \liminf_N \Phi_{m,1}(\sigma_N)$, $\lim_{N\to\infty} \Phi_{m,2}(\sigma_N) = \Phi_{m,2}(\sigma)$. Estimating quantity $|\Phi_{m,2}(J)|$ with the help of [3], Theorem 2, we end up with

$$\Phi_m(\sigma) \leqslant \limsup_N \Phi_m(\sigma_N) = \limsup_N \Psi_m(J_N) = \lim_{N\to\infty} \Psi_m(J_N) = \Psi_m(J).$$

The proof is complete. □

3. Sketch of the proof of Lemma 2.1

We begin with considering expressions $\operatorname{tr}(J^{2k} - J_0^{2k})$, arising in (4). Defining $V = J - J_0 = J(a-1, b)$, we have

$$\operatorname{tr}\left(J^{2k} - J_0^{2k}\right) = \operatorname{tr}\sum_{p=1}^{2k} \sum_{i_1+\cdots+i_p=2k-p} V J_0^{i_1} \cdots V J_0^{i_p}.$$

We prove Lemma 2.1 in two steps. First, we reduce the situation to a commutative one. To do this, we bound expressions $|\operatorname{tr}(V J_0^{i_1} \cdots V J_0^{i_p} - V^p J_0^{2k-p})|$ using properties of the commutator $[V, J_0] = V J_0 - J_0 V$. On the second stage, we exploit specifics of $\Psi_m(J)$ to get straightforward estimates of terms obtained after the "commutation".

Lemma 3.1. *Let $\mathbf{i} = (i_1, \ldots, i_p)$ and $\sum_s i_s = n$. Then*

$$VJ_0^{i_1}\cdots VJ_0^{i_p} = V^pJ_0^n + \sum_{\substack{l_1+l_2+l_3=p,\\ p_1+p_2+p_3=n}} C_{\mathbf{l},\mathbf{p}} J_0^{p_1}V^{l_1}\big[V^{l_2}, J_0^{p_2}\big]V^{l_3}J_0^{p_3} + \sum_k^{M_{\mathbf{i},p}} A_k[V, J_0]B_k[V, J_0]C_k,$$

where $\mathbf{p} = (p_1, p_2, p_3)$, $\mathbf{l} = (l_1, l_2, l_3)$, *and* A_k, B_k, C_k *are some bounded operators.*

This proposition leads to the following lemma.

Lemma 3.2. *Let* $\sum_s i_s = 2k - p$. *We have* $|\operatorname{tr}(VJ_0^{i_1}\ldots VJ_0^{i_p} - V^pJ_0^{2k-p})| \leqslant C_2(\|\partial a\|_2^2 + \|\partial b\|_2^2)$ *with some constant* C_2.

The lemma exactly says that, modulo bounded terms, we may assume operators V and J_0 to commute. Turning back to (4), we see that the problem is reduced to estimating $\Psi'_m(J)$,

$$\Psi'_m(J) = \operatorname{tr}\left\{\sum_{p=1}^{2m} V^pF_p(J_0) - \frac{(2m-1)!!}{(2m)!!}\log(I+\tilde\alpha)\right\}, \tag{6}$$

where $\tilde\alpha = \operatorname{diag}\{\alpha_k\} = A - I$, and

$$F_p(J_0) = \sum_{k=[(p+1)/2]}^{m} \frac{(-1)^{k+1}}{2^{2k+1}k}\widetilde{C}_{2m-1}^{2k-1}C_{2k}^{p}\,J_0^{2k-p}.$$

Here, C_k^p is a usual binomial coefficient. Observe that for $p \geqslant m+1$ we have $|\operatorname{tr}(V^pF_p(J_0))| \leqslant \|F_p(J_0)\|\,\|V^p\|_{S_1} \leqslant C_4(\|a-1\|_{m+1}^{m+1} + \|b\|_{m+1}^{m+1})$, where $\|\cdot\|_{S_1}$ is the norm in the class of nuclear operators. Hence, it remains to bound the first m terms in (6). Set $J_{0,p}$ to be a symmetric matrix with 1's on p-th auxiliary diagonals and 0's elsewhere. Surprisingly, the following lemma holds.

Lemma 3.3. *We have*

$$F_p(J_0) = (-1)^{p+1}\frac{(2m-1)!!}{2p(2m)!!}J_{0,p} + \sum_{s=p+1}^{2m-p} d_{p,s}J_{0,s}$$

for some coefficients $d_{p,s}$ *and* $p = 1, m$.

Combining this with explicit form of V^p and a standard series expansion for $\log(I+\tilde\alpha)$, we get the required bound (5).

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