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Partial Differential Equations

On the distributions of the form $\sum_i (\delta_{p_i} - \delta_{n_i})$

Sur les distributions de la forme $\sum_i (\delta_{p_i} - \delta_{n_i})$

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Abstract

We present some properties of the distributions of the form $T = \sum_i (\delta_{p_i} - \delta_{n_i})$, with $\sum_i d(p_i, n_i) < \infty$, which arise in the 3-d Ginzburg–Landau problem studied by Bourgain, Brezis and Mironescu (C. R. Acad. Sci. Paris, Ser. I 331 (2000) 119–124). We show that there always exists an irreducible representation of T. We also extend a result of Smets (C. R. Acad. Sci. Paris, Ser. I 334 (2002) 371–374) which says that T is a measure iff T can be written as a finite sum of dipoles. ***To cite this article: A.C. Ponce, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

On présente quelques propriétés sur les distributions T de la forme $\sum_i (\delta_{p_i} - \delta_{n_i})$, avec $\sum_i d(p_i, n_i) < \infty$, qui interviennent dans le problème de Ginzburg–Landau en 3-d étudié par Bourgain, Brezis et Mironescu (C. R. Acad. Sci. Paris, Ser. I 331 (2000) 119–124). Même dans un cadre plus général, ces formes linéaires ont toujours une représentation irréductible. Notre approche permet aussi de montrer que T est une mesure si et seulement si T peut être écrite comme une somme finie de masses de Dirac, ce qui généralise un résultat de Smets (C. R. Acad. Sci. Paris, Ser. I 334 (2002) 371–374). ***Pour citer cet article : A.C. Ponce, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

Soit (X, d) un espace métrique complet. Étant donnés $p_i, n_i \in X$, $i \in \mathbb{N}$, tels que $\sum_i d(p_i, n_i) < \infty$, on considère la forme linéaire

$$T = \sum_i (\delta_{p_i} - \delta_{n_i}) \tag{1}$$

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agissant sur l'espace de fonctions lipschitziennes $\mathrm{Lip}(X)$. On définit la norme :

$$\|T\| := \sup_{\substack{\zeta\in\mathrm{Lip}(X)\\ |\zeta|_{\mathrm{Lip}}\leqslant 1}} \langle T,\zeta\rangle = \sup_{\substack{\zeta\in\mathrm{Lip}(X)\\ |\zeta|_{\mathrm{Lip}}\leqslant 1}} \sum_i \bigl[\zeta(p_i)-\zeta(n_i)\bigr], \tag{2}$$

où on dénote par $|\zeta|_{\mathrm{Lip}}$ la meilleure constante de Lipschitz de ζ.

On sait que (voir [2, Lemme 12′])

$$\|T\| = \inf_{(\tilde p_i),(\tilde n_i)} \biggl\{ \sum_i d(\tilde p_i,\tilde n_i)\colon\ T=\sum_i (\delta_{\tilde p_i}-\delta_{\tilde n_i}) \text{ dans } \bigl[\mathrm{Lip}(X)\bigr]' \biggr\}, \tag{3}$$

ce qui justifie l'appellation de longueur de la connexion minimale pour $\|T\|$.

Alors que le supremum dans la définition (2) est toujours atteint, cela n'est pas nécessairement le cas pour l'infimum dans l'expression (3) (voir Exemple 1). En revanche, on prouve le résultat suivant :

Théorème 0.1. *Si $\overline{\bigcup_i \{p_i\}\cup\bigcup_i \{n_i\}}$ est dénombrable, alors l'infimum dans l'expression* (3) *est atteint. Plus généralement, la même conclusion reste valable si l'on suppose $\mathcal{H}^1\bigl(\overline{\bigcup_i \{p_i\}\cup\bigcup_i \{n_i\}}\bigr)=0$, où $\mathcal{H}^1$ désigne la* 1*-mesure de Hausdorff.*

La démonstration de ce théorème repose sur l'existence de représentations irréductibles de T. On observe d'abord que T peut être écrite de plusieurs façons comme une somme infinie de masses de Dirac. Alors, on dit que la représentation de T en (1) n'est *pas* irréductible si et seulement si l'une des deux conditions suivantes est satisfaite : (a) il existe $i,j\geqslant 1$ tel que $p_i=n_j$; (b) il existe un ensemble infini $\widetilde{\mathbb{N}}\subset\mathbb{N}$ et $r,q\in X$ tels que $\sum_{i\in\widetilde{\mathbb{N}}}(\delta_{p_i}-\delta_{n_i})=\delta_r-\delta_q$ dans $[\mathrm{Lip}(X)]'$.

On a le théorème suivant :

Théorème 0.2. *T admet toujours une représentation irréductible.*

Enfin, on considère le cas où T est une mesure, i.e. on suppose qu'il existe $C>0$ tel que $|\langle T,\zeta\rangle|\leqslant C\|\zeta\|_\infty$ $\forall\zeta\in\mathrm{BLip}(X)$, l'espace des fonctions lipschitziennes bornées. En utilisant le Théorème 0.2, on démontre le

Théorème 0.3. *T est une mesure si et seulement si il existe $a_1,\dots,a_k\in X$ et $d_1,\dots,d_k\in\mathbb{Z}$, avec $\sum_i d_i=0$, tels que $T=\sum_{i=1}^k d_i\delta_{a_i}$ dans* $[\mathrm{Lip}(X)]'$.

Ce résultat avait été établi par Smets [7] pour des espaces métriques complets et localement compacts.

Les démonstrations détaillées sont présentées dans [6].

1. Introduction

Given a complete metric space (X,d), we consider the distributions T of the form $\sum_i(\delta_{p_i}-\delta_{n_i})$, with $\sum_i d(p_i,n_i)<\infty$, given by (7). Roughly speaking, T describes the location and the topological degree of classes of maps u defined on X. Here are two examples of application:

(a) $X=\mathbb{R}^3$ and $u\in H^1(\mathbb{R}^3;S^2)$, which arises in the study of liquid crystals (see [4]);
(b) $X=S^2$ and $u\in H^{1/2}(S^2;S^1)$, which is related to the Ginzburg–Landau problem in $B_1\subset\mathbb{R}^3$ (see [1,2]); here we consider S^2 equipped with the Euclidean metric induced by $\mathbb{R}^3$, although the geodesic distance in S^2 is also of interest.

In this paper, we present some properties of T. Our proofs rely on the existence of irreducible representations of T, a notion which we introduce in Section 4.

2. Minimal connections

Let X be a complete metric space. Given a finite number of points (not necessarily distinct) $p_1, \ldots, p_k$, $n_1, \ldots, n_k$ in X, the length of the minimal connection between these points is given by (see [5])

$$L := \min_{\sigma \in S_k} \sum_{i=1}^{k} d(p_i, n_{\sigma(i)}), \tag{4}$$

where S_k denotes the group of permutations of $\{1, \ldots, k\}$. It can be shown that the number L satisfies (see [5]; see also [3] for an elementary proof)

$$L = \sup_{|\zeta|_{\mathrm{Lip}} \leqslant 1} \sum_{i=1}^{k} \big[\zeta(p_i) - \zeta(n_i)\big], \tag{5}$$

where $|\zeta|_{\mathrm{Lip}}$ denotes the best Lipschitz constant of ζ. Moreover, the supremum in (5) is achieved.

More generally, consider two sequences $(p_i), (n_i) \subset X$ such that $\sum_i d(p_i, n_i) < \infty$. We now introduce the following linear form acting on $\mathrm{Lip}(X)$ (the vector space of Lipschitz functions in X)

$$T := \sum_i (\delta_{p_i} - \delta_{n_i}) \tag{6}$$

given by

$$\langle T, \zeta\rangle = \sum_i \big[\zeta(p_i) - \zeta(n_i)\big] \quad \forall \zeta \in \mathrm{Lip}(X). \tag{7}$$

Note that T is well-defined and continuous in $\mathrm{Lip}(X)$. In other words, $T \in [\mathrm{Lip}(X)]'$.

Motivated by (5), we define the length of the minimal connection between these points as (see [2])

$$\|T\| := \sup_{|\zeta|_{\mathrm{Lip}} \leqslant 1} \langle T, \zeta\rangle = \sup_{|\zeta|_{\mathrm{Lip}} \leqslant 1} \sum_i \big[\zeta(p_i) - \zeta(n_i)\big]. \tag{8}$$

We point out that the supremum is still achieved in this case. In [6] we compare this number with other possible definitions.

Let

$$\mathcal{Z} := \big\{T \in \big[\mathrm{Lip}(X)\big]' \colon T \text{ has the form (6) for some } (p_i), (n_i) \subset X \text{ such that } \textstyle\sum_i d(p_i, n_i) < \infty\big\}. \tag{9}$$

Note that if $T \in \mathcal{Z}$ then $-T \in \mathcal{Z}$, and $T_1 + T_2 \in \mathcal{Z}$ whenever $T_1, T_2 \in \mathcal{Z}$. Moreover, $\|\cdot\|$ induces a metric in $\mathcal{Z}$.

There are infinitely many possible representations of $T \in \mathcal{Z}$ as a sum of the form (6). Moreover, such representations need not be equivalent modulo a permutation of points. In fact, if (q_i) is a sequence rapidly converging to p in X (in the sense that $\sum_i d(q_i, q_{i+1}) < \infty$), then we can write $\delta_p - \delta_n = \sum_{i=1}^{\infty} (\delta_{q_{i+1}} - \delta_{q_i})$ in $[\mathrm{Lip}(X)]'$, where $n := q_1$.

The next proposition extends (4) to our more general setting (see [2, Lemma 12′])

Proposition 2.1. *For any $T \in \mathcal{Z}$ we have*

$$\|T\| = \inf_{(\tilde{p}_i), (\tilde{n}_i)} \left\{\sum_i d(\tilde{p}_i, \tilde{n}_i) \colon T = \sum_i (\delta_{\tilde{p}_i} - \delta_{\tilde{n}_i}) \text{ in } \big[\mathrm{Lip}(X)\big]'\right\}. \tag{10}$$

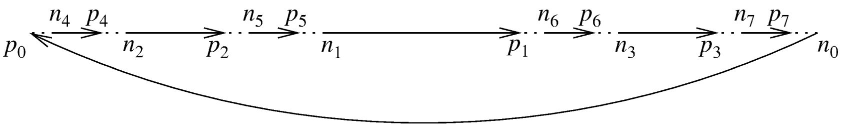

Fig. 1. Dipoles $\delta_{p_i} - \delta_{n_i}$ in Example 1.
Fig. 1. Dipoles $\delta_{p_i} - \delta_{n_i}$ de l'Exemple 1.

3. On the existence of minimal connections

We first introduce the notion of support of T:

Definition 3.1. Let $(\omega_i)_{i\in I}$ be the family of open subsets of X such that, for each $i \in I$, the following holds: if $\zeta \in \mathrm{Lip}(X)$ and $\zeta \equiv 0$ on $X\backslash\omega_i$, then $\langle T, \zeta\rangle = 0$. We set $\operatorname{supp} T := X \backslash \bigcup_{i\in I} \omega_i$.

Clearly, $\operatorname{supp} T \subset \overline{\bigcup_i \{p_i\} \cup \bigcup_i \{n_i\}}$, although the strict inequality can actually occur (see, however, Theorem 4.5 below).

In contrast with the case of a finite number of points, the infimum in (10) may not be achieved in general (see Example 1). Here is a case where it is still attained:

Theorem 3.2. *If* $\mathcal{H}^1(\operatorname{supp} T) = 0$, *then the infimum in* (10) *is attained. In other words, there exist* $(\tilde{p}_i)$, $(\tilde{n}_i)$ *in* X *such that* $\|T\| = \sum_i d(\tilde{p}_i, \tilde{n}_i)$ *and* $T = \sum_i (\delta_{\tilde{p}_i} - \delta_{\tilde{n}_i})$.

Above, $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure. In particular, if $\overline{\bigcup_i \{p_i\} \cup \bigcup_i \{n_i\}}$ is countable, then Theorem 3.2 holds.

In general, it is still possible to decompose T in terms of simpler functionals, taking into account the length of its minimal connection. But let us first make another definition:

Definition 3.3. $T \in \mathcal{Z}$ is *regular* if there exist $(\tilde{p}_i), (\tilde{n}_i) \subset X$ such that $T = \sum_i (\delta_{\tilde{p}_i} - \delta_{\tilde{n}_i})$ and $\|T\| = \sum_i d(\tilde{p}_i, \tilde{n}_i)$. $T \in \mathcal{Z}$ is *singular* if whenever $T = T_1 + T_2$, $\|T\| = \|T_1\| + \|T_2\|$ and T_1 is regular, then $T_1 = 0$.

Here is an example of $T \in \mathcal{Z}$ which is singular:

Example 1. Let $X = [0, 1]$ and $C_\alpha \subset [0, 1]$ be a Cantor-type set with Lebesgue measure $\alpha \in (0, 1)$. We denote by $(J_k)_{k\geqslant 1}$, $J_k = (n_k, p_k)$, the sequence of disjoint open intervals which are removed from $[0, 1]$ in the construction of C_α. We then take $p_0 = 0$ and $n_0 = 1$. It can be shown (see [6]) that $T = \sum_{i\geqslant 0} (\delta_{p_i} - \delta_{n_i})$ is singular and $\|T\| = \alpha$. Note that the length of the minimal connection of T is actually carried by the set $C_\alpha = \operatorname{supp} T$. For descriptive purposes we can think of representing each dipole $\delta_{p_i} - \delta_{n_i}$ as an arrow from n_i to p_i. In Fig. 1 we represent T geometrically according to this convention.

We have the following (the geometric meaning of Theorem 3.4 is explained in [6])

Theorem 3.4. *For any* $T \in \mathcal{Z}$ *there exist* $T_{\mathrm{reg}}, T_{\mathrm{sing}} \in \mathcal{Z}$ *such that* T_{reg} *is regular,* T_{sing} *is singular,*

$$T = T_{\mathrm{reg}} + T_{\mathrm{sing}} \quad \textit{and} \quad \|T\| = \|T_{\mathrm{reg}}\| + \|T_{\mathrm{sing}}\|. \tag{11}$$

Moreover, there exists $(T_k) \subset \mathcal{Z}$ *such that*

$$T_{\mathrm{sing}} = \sum_k T_k, \quad \|T_{\mathrm{sing}}\| = \sum_k \|T_k\| \quad \textit{and} \quad \|T_k\| = \mathcal{H}^1(\operatorname{supp} T_k) \quad \forall k. \tag{12}$$

Each $\operatorname{supp} T_k$ *is homeomorphic to the Cantor set in* $\mathbb{R}$.

The decomposition of T in terms of a regular and a singular part, as in (11), need not be unique (see [6]).

We point out that Theorem 3.2 is a special case of the above. In fact, it follows from the proof of Theorem 3.4 that $\operatorname{supp} T = \operatorname{supp} T_{\mathrm{reg}} \cup \operatorname{supp} T_{\mathrm{sing}}$ and $\bigcup_k \operatorname{supp} T_k \subset \operatorname{supp} T_{\mathrm{sing}}$. Therefore, if $\mathcal{H}^1(\operatorname{supp} T) = 0$, then $\|T_k\| = 0$ for each k. We conclude that $T = T_{\mathrm{reg}}$ is regular.

4. Irreducible representations

A natural question regarding $T \in \mathcal{Z}$ is whether it has a "simplest" representation in the following sense:

Definition 4.1. The representation $\sum_i (\delta_{p_i} - \delta_{n_i})$ is *reducible* if there exist $\mathbb{N}_1 \subset \mathbb{N}_2 \subset \mathbb{N}$, $\operatorname{card} \mathbb{N}_1 < \operatorname{card} \mathbb{N}_2$, and points $r_i, q_i \in X$, $i \in \mathbb{N}_1$, such that

$$\sum_{i \in \mathbb{N}_2} (\delta_{p_i} - \delta_{n_i}) = \sum_{i \in \mathbb{N}_1} (\delta_{r_i} - \delta_{q_i}) \quad \text{in } \big[\mathrm{Lip}(X)\big]'. \tag{13}$$

$\sum_i (\delta_{p_i} - \delta_{n_i})$ will be called *irreducible* if it is not reducible.

The next result states that one can always find an irreducible representation of T:

Theorem 4.2. *Any linear form $T \in \mathcal{Z}$ has an irreducible representation. More precisely, there exist sequences $(\hat{p}_i)$, $(\hat{n}_i)$ in X such that $\sum_i d(\hat{p}_i, \hat{n}_i) < \infty$,*

$$T = \sum_i (\delta_{\hat{p}_i} - \delta_{\hat{n}_i}) \quad \textit{in } \big[\mathrm{Lip}(X)\big]', \tag{14}$$

and this representation is irreducible.

Our proof of Theorem 4.2 relies on the notion of maximal paths. This approach requires the following lemma which is interesting in its own:

Lemma 4.3. *If*

$$\sum_i (\delta_{p_i} - \delta_{n_i}) = (\delta_{r_1} - \delta_{q_1}) + (\delta_{r_2} - \delta_{q_2}) \quad \textit{in } \big[\mathrm{Lip}(X)\big]' \tag{15}$$

for some $r_1, q_1, r_2, q_2 \in X$, then there exists $\widetilde{\mathbb{N}} \subset \mathbb{N}$ such that

$$\sum_{i \in \widetilde{\mathbb{N}}} (\delta_{p_i} - \delta_{n_i}) \quad \textit{equals} \quad (\delta_{r_1} - \delta_{q_1}) \quad \textit{or} \quad (\delta_{r_1} - \delta_{q_2}) \quad \textit{in } \big[\mathrm{Lip}(X)\big]'. \tag{16}$$

A simple consequence of this lemma is the corollary below which makes our notion of irreducible representations much easier to handle:

Corollary 4.4. $\sum_i (\delta_{p_i} - \delta_{n_i})$ *is reducible if, and only if, one of the following conditions holds*:

(a) $p_i = n_j$ *for some* $i, j \geqslant 1$;
(b) *there exists an infinite set* $\widetilde{\mathbb{N}} \subset \mathbb{N}$ *and* $r, q \in X$ *such that*

$$\sum_{i \in \widetilde{\mathbb{N}}} (\delta_{p_i} - \delta_{n_i}) = \delta_r - \delta_q \quad \textit{in } \big[\mathrm{Lip}(X)\big]'. \tag{17}$$

If T can be written as a finite sum of dipoles of the form $\delta_p - \delta_n$, then the irreducible representation of T is unique (modulo a permutation of the points). This may not be the case in general. Assume, for example, that $X = [0,1]$, and let (p_i), (n_i) be two sequences converging to 0 such that $p_i > n_i > p_{i+1}$ for every $i \geqslant 1$. Then

$$\sum_{i=1}^{\infty}(\delta_{p_i} - \delta_{n_i}), \quad (\delta_{p_1} - \delta_0) + \sum_{i=1}^{\infty}(\delta_{p_{i+1}} - \delta_{n_i}), \quad (\delta_{p_1} - \delta_0) + (\delta_{p_2} - \delta_0) + \sum_{i=1}^{\infty}(\delta_{p_{i+2}} - \delta_{n_i}), \ldots$$

are all irreducible representations of the same operator in $[\mathrm{Lip}([0,1])]'$.

However, we have the following

Theorem 4.5. *Assume* (14) *is an irreducible representation of T. Then*

$$\operatorname{supp} T = \overline{\bigcup_i \{\hat{p}_i\} \cup \bigcup_i \{\hat{n}_i\}}. \tag{18}$$

In particular, if $\zeta \in \mathrm{Lip}(X)$ and $\zeta \equiv 0$ on $\operatorname{supp} T$, then $\langle T, \zeta\rangle = 0$.

5. Characterizations of T as a measure

A simple consequence of Theorem 4.2 is the corollary below:

Corollary 5.1. *Let $T \in \mathcal{Z}$. If $\operatorname{supp} T$ is finite, then there exist finitely many points $\hat{p}_1, \ldots, \hat{p}_{k_0}, \hat{n}_1, \ldots, \hat{n}_{k_0} \in X$ such that $T = \sum_{i=1}^{k_0}(\delta_{\hat{p}_i} - \delta_{\hat{n}_i})$ in $[\mathrm{Lip}(X)]'$.*

Another result in this direction is the theorem below which completely solves an open problem raised by H. Brezis. Here, we denote by $\mathrm{BLip}(X)$ the subspace of bounded Lipschitz functions:

Theorem 5.2. *Let $T \in \mathcal{Z}$. Assume that $|\langle T, \zeta\rangle| \leqslant C\|\zeta\|_\infty$ $\forall \zeta \in \mathrm{BLip}(X)$ for some $C > 0$. Then there exist finitely many points $a_1, \ldots, a_k$ and integers $d_1, \ldots, d_k$, $\sum_i d_i = 0$, such that $T = \sum_{i=1}^k d_i \delta_{a_i}$ in $[\mathrm{Lip}(X)]'$.*

We point out that the conclusion of both results is the same (since $\sum_i d_i = 0$). Our proof of Theorem 5.2 makes use of the existence of irreducible representations of T. Theorem 5.2 has been proved by Smets [7] (using the Riesz Representation Theorem) under the additional assumption that X is locally compact.

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