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C. R. Acad. Sci. Paris, Ser. I 336 (2003) 677–680
COMPTES RENDUS MATHEMATIQUE

Numerical Analysis

Uniformly stable preconditioned mixed boundary element method for low-frequency electromagnetic scattering

Méthode d'éléments finis de frontière mixte préconditionnée uniformément stable pour la diffraction d'ondes électromagnétiques à basse fréquence

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Received 21 October 2002; accepted 11 March 2003

Presented by Olivier Pironneau

Abstract

We propose a mixed boundary finite element discretization of the Electric Field Integral Equation for which we have an Inf–Sup condition which is uniform in both the mesh-width h and the wave-number k, for small enough h and k. For this equation we construct a preconditioner such that the spectral condition number of the preconditioned system is also bounded independently of k and h. ***To cite this article: S.H. Christiansen, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Nous proposons une méthode d'éléments finis frontière mixtes pour l'équation intégrale du champ électrique pour laquelle nous démontrons une condition Inf–Sup uniforme par rapport au pas du maillage h et au nombre d'onde k, pour h et k suffisamment petits. Pour cette équation nous construisons un préconditionneur tel que le conditionnement spectral du système préconditionné soit borné indépendamment de h et k. ***Pour citer cet article : S.H. Christiansen, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

1. The continuous problem

Let Ω_- be a bounded domain in $\mathbb{R}^3$ with a smooth boundary Γ. The exterior domain $\mathbb{R}^3 \setminus (\Omega_- \cup \Gamma)$ is denoted Ω_+ and the outward normal on Γ is denoted n. The tangential trace operator is denoted γ_{T} and the normal trace operator is denoted γ_{n}.

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[1] This work was carried out while the author was a Post-Doc visitor at the Seminar for Applied Mathematics at ETH, Zürich.

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doi:10.1016/S1631-073X(03)00156-0

Let Z be a positive constant, called impedance. For each wavenumber $k > 0$ the time-harmonic Maxwell equations (in any given open region of $\mathbb{R}^3$) are:

$$\operatorname{curl} E = +\mathrm{i}kZH, \qquad \operatorname{curl} H = -\mathrm{i}k/ZE. \tag{1}$$

Given a family $(E_k^{\mathrm{inc}}, H_k^{\mathrm{inc}})$ for small positive k of solutions of Maxwell's equations on a neighborhood of Γ we are interested, for each k, in the solution (E_k, H_k) of Maxwell's equations in Ω_- or Ω_+ satisfying the perfect conductor boundary condition $\gamma_{\mathrm{T}} E_k = -\gamma_{\mathrm{T}} E_k^{\mathrm{inc}}$, and (in the exterior domain) the Silver–Müller radiation condition.

We use potentials to represent E_k. Let G_k denote the standard Green kernel of $-\Delta - k^2$ and let Φ_k be the single layer potential defined on scalar or tangent fields u on Γ by:

$$(\Phi_k u)(y) = \int_\Gamma G_k(x, y) u(x)\, \mathrm{d}x, \quad G_k(x, y) = \frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{4\pi|x-y|}. \tag{2}$$

We represent E_k as an electric field generated by a tangent field u_k on Γ (the electric current). More precisely we put $E_k(y) = (\operatorname{grad}\operatorname{div} + k^2)(\Phi_k u_k)$. Letting $A_k = -\gamma_{\mathrm{T}}(\operatorname{grad}\operatorname{div} + k^2)\Phi_k$, the problem is to solve the Electric Field Integral Equation (EFIE) $A_k u_k = \gamma_{\mathrm{T}} E_k^{\mathrm{inc}}$.

The operator A_k is continuous from $X = \mathrm{H}_{\mathrm{div}}^{-1/2}(\Gamma)$ to its dual $X' = \mathrm{H}_{\mathrm{rot}}^{-1/2}(\Gamma)$ (see, e.g., [5]), and the EFIE can be put in variational form:

$$u_k \in X,\ \forall u' \in X, \quad \langle A_k u_k, u'\rangle = \langle E_k^{\mathrm{inc}}, u'\rangle. \tag{3}$$

We denote by a_k the associated bilinear form; its expression on smooth fields is:

$$a_k(u, v) = \iint_{\Gamma\times\Gamma} G_k(x, y)\big(\operatorname{div} u(x)\operatorname{div} v(y) - k^2 u(x)\cdot v(y)\big)\, \mathrm{d}x\, \mathrm{d}y. \tag{4}$$

Following Bendali [1] this variational problem is solved with the Galerkin method on div-conforming Finite Element spaces on the boundary. At low frequencies one sees that the problem is that the limit of the operator A_k as $k \to 0$, is degenerated; in fact the limit is not even Fredholm since its kernel contains the infinite dimensional space $\operatorname{rot}\mathrm{H}^{1/2}(\Gamma)$.

The object of this paper is to compute approximations of u_k in a stable way for small k.

2. The continuous remedy

For simplicity we suppose that Γ is connected and simply connected. As remarked by de La Bourdonnaye [2], if we put $V = \operatorname{grad}\mathrm{H}^{3/2}(\Gamma)$ and $W = \operatorname{rot}\mathrm{H}^{1/2}(\Gamma)$, then V and W are closed in X and we have the decomposition:

$$X = V \oplus W. \tag{5}$$

We put $S = \mathrm{H}^{1/2}(\Gamma)$, and for any space Y of scalar fields on Γ we put $Y^\bullet = \{u \in Y\colon \langle u, 1\rangle = 0\}$. Also, for any Hilbert space Y, $Y^\star$ denotes the Hilbert space of continuous linear forms on Y.

Let $\Xi_k : V \times S^\bullet \to X$ denote the isomorphism defined by $\Xi_k(v, p) = v + k^{-1}\operatorname{rot} p$. The four blocks of the bilinear form $\tilde{a}_k$ on $V \times S^\bullet$ defined by $\tilde{a}_k((v, p), (v', p')) = a_k(\Xi_k(v, p), \Xi_k(v', p'))$, have the expression:

$$\begin{pmatrix} \iint G_k(x, y)(\operatorname{div} v(x)\operatorname{div} v'(y) - k^2 v(x)\cdot v'(y))\, \mathrm{d}x\, \mathrm{d}y & -k\iint G_k(x, y)\operatorname{rot} p(x)\cdot v'(y)\, \mathrm{d}x\, \mathrm{d}y \\ -k\iint G_k(x, y) v(x)\cdot \operatorname{rot} p'(y)\, \mathrm{d}x\, \mathrm{d}y & -\iint G_k(x, y)\operatorname{rot} p(x)\cdot\operatorname{rot} p'(y)\, \mathrm{d}x\, \mathrm{d}y \end{pmatrix}. \tag{6}$$

Since there is $C > 0$ such that

$$\forall v \in V \quad \|v\|_X \leqslant C\|\operatorname{div} v\|_{\mathrm{H}^{-1/2}(\Gamma)}, \qquad \forall p \in S^\bullet \quad \|p\|_S \leqslant C\|\operatorname{rot} p\|_{\mathrm{H}_{\mathrm{T}}^{-1/2}(\Gamma)}, \tag{7}$$

the two diagonal blocks are coercive hence invertible for $k = 0$. We remark also that the coupling blocks vanish for $k = 0$. Concerning the right-hand sides we remark that

$$k^{-1}\langle \gamma_{\mathrm{T}} E_k^{\mathrm{inc}}, \operatorname{rot} p'\rangle = \mathrm{i}Z\langle \gamma_{\mathrm{n}} H_k^{\mathrm{inc}}, p'\rangle. \tag{8}$$

If for instance the family of incident waves consists of plane waves:

$$E_k^{\mathrm{inc}}(x) = E_0\,\mathrm{e}^{\mathrm{i}k\sigma\cdot x}, \qquad H_k^{\mathrm{inc}}(x) = 1/Z E_0 \times \sigma\,\mathrm{e}^{\mathrm{i}k\sigma\cdot x}, \tag{9}$$

then the limit of $\gamma_{\mathrm{T}} E_k^{\mathrm{inc}}$ is a surface gradient and $\gamma_{\mathrm{n}} H_k^{\mathrm{inc}}$ has a non-zero limit. Thus both components of the right-hand sides considered as elements of $V^\star \times S^{\bullet\star}$ have a non-zero limit as $k \to 0$. It follows that with the decomposition $u_k = \Xi_k(v_k, p_k)$ both v_k and p_k have a non-zero limit as $k \to 0$.

We now turn to the preconditioning of the variational problem associated with (6) and we recall the remark made in [3,4] that a preconditioner is obtained by an invertible bilinear form on a dual space. Since the off-diagonal terms are small in norm and compact it is enough to precondition the two diagonal blocks.

For the first block, we proceed as follows. Put $V' = \operatorname{grad} \mathrm{H}^{1/2}(\Gamma)$. Then we remark that the $\mathrm{L}^2_{\mathrm{T}}(\Gamma)$-bilinear form extends continuously to an invertible bilinear form on $V' \times V$. Let $\Theta_1 : V^\star \to V'$ be the corresponding isomorphism. We remark furthermore that V' is a closed subspace of $\mathrm{H}_{\mathrm{T}}^{-1/2}(\Gamma)$, hence we can use the bilinear form associated with the single layer operator on tangent fields as a preconditioner.

For the second block, the induced operator on $S^\bullet \to S^{\bullet\star}$ is the main part of the hypersingular operator appearing in acoustics. It can be efficiently preconditioned by the single layer operator [6,3]. As a matter of notations we put $S' = \mathrm{H}^{-1/2}(\Gamma)$ so that the $\mathrm{L}^2(\Gamma)$-bilinear form extends continuously to an invertible bilinear form on $S'^\bullet \times S^\bullet$, and let $\Theta_2 : S^{\bullet\star} \to S'^\bullet$ be the corresponding isomorphism.

Thus, on $V' \times S'^\bullet$ we use the bilinear form b whose block expression is:

$$\begin{pmatrix} \iint G_0(x,y) v(x)\cdot v'(y)\,\mathrm{d}x\,\mathrm{d}y & 0 \\ 0 & -\iint G_0(x,y) q(x) q'(y)\,\mathrm{d}x\,\mathrm{d}y \end{pmatrix}. \tag{10}$$

Letting $\Theta : V^\star \times S^{\bullet\star} \to V' \times S'^\bullet$ be the map componentwise induced by Θ_1 and Θ_2, and associating an operator $\tilde{\mathcal{A}}_k : V \times S^\bullet \to V^\star \times S^{\bullet\star}$ with $\tilde a_k$ and $\mathcal{B} : V' \times S'^\bullet \to V'^\star \times S'^{\bullet\star}$ with b we have:

Proposition 2.1. *There exists $\varepsilon > 0$ such that for all $k \in [0, \varepsilon]$ the operator $\Theta^\star \mathcal{B} \Theta \tilde{\mathcal{A}}_k$ is an automorphism of $V \times S^\bullet$ and all terms of the composition are isomorphisms whose norm and norm of the inverse are bounded independently of k in* $[0, \varepsilon]$.

3. Discretization

Since $V = \operatorname{grad} \mathrm{H}^{3/2}(\Gamma)$ it would be cumbersome to implement a conforming finite element discretization of the variational problem on $V \times S^\bullet$. Instead we propose the following non-conforming method.

Suppose we have (finite dimensional) subspaces X_h of $X \cap \mathrm{H}^0_{\mathrm{div}}(\Gamma)$ and S_h of $S \cap \mathrm{H}^1(\Gamma)$, which are stable under complex conjugation, which are such that S_h contains the constant fields and we have an exact sequence:

$$S_h \xrightarrow{\mathrm{rot}} X_h \xrightarrow{\mathrm{div}} \mathrm{L}^2(\Gamma). \tag{11}$$

We define V_h by:

$$V_h = \big\{ u \in X_h \colon \forall p \in S_h \quad \langle u, \operatorname{rot} p\rangle = 0 \big\}. \tag{12}$$

We keep the notation $\tilde a_k$ to denote the extension of $\tilde a_k$ to $X \times S$ whose block-wise expression is given by (6). We solve the system: Find $(v_{kh}, p_{kh}) \in V_h \times S_h^\bullet$, such that for all $(v', p') \in V_h \times S_h^\bullet$; we have:

$$\tilde a_k\big((v_{kh}, p_{kh}), (v', p')\big) = \big\langle E_k^{\mathrm{inc}}, v'\big\rangle + \mathrm{i}Z\big\langle H_k^{\mathrm{inc}} \cdot n, p'\big\rangle. \tag{13}$$

Recall the definition of the gap: $\delta_X(V_h, V) = \sup_{v_h \in V_h} \inf_{v \in V} \|v_h - v\|_X / \|v_h\|_X$. Our first proposition concerns the well posedness of the discrete system.

Proposition 3.1. *If $\delta_X(V_h, V) \to 0$ as $h \to 0$ then there exists $\varepsilon > 0$, $h_0 > 0$ and $C > 0$ such that for all $k \in [0, \varepsilon]$, all $h < h_0$ we have*:

$$\inf_{(v,p)\in V_h \times S_h^\bullet} \sup_{(v',p')\in V_h \times S_h^\bullet} \frac{|\tilde a_k((v,p),(v',p'))|}{\|(v,p)\|_{X\times S} \|(v',p')\|_{X\times S}} \geqslant \frac{1}{C}. \tag{14}$$

In general we do not have a basis of V_h, hence solving this system requires some extra work. In our case this will be carried out by the preconditioner which we define now. It should be checked that in what follows only bases of X_h and S_h are needed.

Let $\Theta_{1h} : X_h^\star \to X_h$ denote the map which to any $\ell \in X_h^\star$ associates the solution of:

$$v \in V_h,\ \forall v' \in V_h \quad \langle v, v'\rangle = \ell(v'). \tag{15}$$

For $\ell \in X_h^\star$, $\Theta_{1h}\ell$ can be computed simply as the solution u of:

$$p \in S_h^\bullet,\ \forall p' \in S_h^\bullet \quad \langle \operatorname{rot} p, \operatorname{rot} p'\rangle = \ell(\operatorname{rot} p'), \qquad u \in X_h,\ \forall u' \in X_h \quad \langle u, u'\rangle = \ell(u') - \langle \operatorname{rot} p, u'\rangle. \tag{16}$$

We define the discretization Θ_{2h} of Θ_2 to be the map which to $\ell \in S_h^\star$ associates the solution p of:

$$p \in S_h^\bullet,\ \forall p' \in S_h^\bullet \quad \langle p, p'\rangle = \ell(p'). \tag{17}$$

Let $\Theta_h : X_h^\star \times S_h^\star \to X_h \times S_h$ be the association of Θ_{1h} and Θ_{2h}. We keep the notation b for the extension of b from $V' \times S'^\bullet$ to $\mathrm{H}_\mathrm{T}^{-1/2}(\Gamma) \times S'$ keeping the block expression (10). Let $\mathcal{B}_h : X_h \times S_h \to X_h^\star \times S_h^\star$ be the map induced by b. We also denote by $\tilde{\mathcal{A}}_{kh}$ the map induced by $\tilde{a}_k$ on $X_h \times S_h \to X_h^\star \times S_h^\star$.

One sees that the operator $\Theta_h^\star \mathcal{B}_h \Theta_h$ is a surjection onto $V_h \times S_h^\bullet$. For $\ell \in (X_h \times S_h^\bullet)^\star$, $\Theta_h^\star \mathcal{B}_h \Theta_h \ell$ depends only on $\ell|_{V_h \times S_h^\bullet}$. It follows that the conjugate gradient algorithm for $\tilde{\mathcal{A}}_{kh}$ on $X_h \times S_h^\bullet$, preconditioned by $\Theta_h^\star \mathcal{B}_h \Theta_h$ yields iterates in $V_h \times S_h^\bullet$ converging to the solution of (13). Morevover $\Theta_h^\star \mathcal{B}_h \Theta_h \tilde{\mathcal{A}}_{kh}$ determines a bijection $V_h \times S_h^\bullet \to V_h \times S_h^\bullet$ (denoted Λ_{kh} in what follows) whose spectral condition number κ_{kh} is bounded independently of k in an interval $[0, \varepsilon]$.

More precise estimates on κ_{kh} and the convergence of Krylov subspace methods, depend on the actual Galerkin spaces. Examples of finite element spaces which satisfy the above conditions include the case where we have quasi-uniform triangulations of Γ and take for X_h Raviart–Thomas vector FE of degree n and for S_h the scalar continuous piecewise P^{n+1} FE. Then we also have the following stability property:

Proposition 3.2. *There exists $\varepsilon > 0$, $h_0 > 0$ and $C_1, C_2 > 0$ such that for all $k \in [0, \varepsilon]$, all $h < h_0$ we have for all $(u, p) \in V_h \times S_h^\bullet$:*

$$\|\Lambda_{kh}(u,p)\|_0 \leqslant C_1 \|(u,p)\|_0 \quad \textit{with } \|(u,p)\|_0^2 = \|u\|^2_{\mathrm{H}^0_{\mathrm{div}}(\Gamma)} + \|p\|^2_{\mathrm{H}^1(\Gamma)}, \tag{18}$$

$$\|\Lambda_{kh}(u,p)\|_{-1/2} \geqslant C_2^{-1} \|(u,p)\|_{-1/2} \quad \textit{with } \|(u,p)\|_{-1/2}^2 = \|u\|^2_{\mathrm{H}^{-1/2}_{\mathrm{div}}(\Gamma)} + \|p\|^2_{\mathrm{H}^{1/2}(\Gamma)}. \tag{19}$$

It follows that for (k, h) in the range $[0, \varepsilon] \times]0, h_0[$, the spectral radius of Λ_{kh} is smaller than C_1, while that of its inverse is smaller than C_2, hence κ_{kh} is bounded by $C_1 C_2$.

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