

## Dynamical Systems

# A remark about hyperbolic infranilautomorphisms 

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#### Abstract

We show that any exact 2 -form, preserved by a hyperbolic infranilautomorphism, must be zero. We then deduce two propositions about geometric Anosov flows and the time change of suspensions. To cite this article: Y. Fang, C. R. Acad. Sci. Paris, Ser. I 336 (2003). © 2003 Académie des sciences. Published by Éditions scientifiques et médicales Elsevier SAS. All rights reserved.


## Résumé

Une remarque sur les infranilautomorphismes hyperboliques. Nous montrons qu'une 2-forme exacte, préservée par un infranilautomorphisme hyperbolique, s'annule, et nous en déduisons deux propositions sur les flots d'Anosov géométriques et le changement du temps des suspensions. Pour citer cet article: Y. Fang, C. R. Acad. Sci. Paris, Ser. I 336 (2003). © 2003 Académie des sciences. Published by Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

## 1. Introduction

Let $M$ be $C^{\infty}$-closed manifold. A diffeomorphism $\phi$ of $M$ is called Anosov, if there exists a $\phi$-invariant splitting $T M=E^{+} \oplus E^{-}$and two positive numbers $a$ and $b$, such that

$$
\forall u^{ \pm} \in E^{ \pm}, \forall n \in \mathbb{Z}^{+}, \quad\left\|D \phi^{\mp n}\left(u^{ \pm}\right)\right\| \leqslant c \mathrm{e}^{-b n}\left\|u^{ \pm}\right\|
$$

where the distributions $E^{-}$and $E^{+}$are called stable and unstable. If $E^{-}$and $E^{+}$are $C^{\infty}$ subbundles, then $\phi$ is said to have smooth distributions. Anosov flows and the corresponding notions are defined similarly (see [6]). The known examples of Anosov diffeomorphisms with smooth distributions are the hyperbolic infranilautomorphisms on some infranilmanifolds. Such diffeomorphisms are finitely covered by the hyperbolic nilautomorphisms, which are simply constructed as follows (see [3] for more details).

Let $N$ be a connected and simply connected nilpotent Lie group and $\Gamma$ be a lattice in $N$. Let $\psi$ be an automorphism of $N$ such that $\psi(\Gamma)=\Gamma$. If no eigenvalue of the differential $D_{e} \psi$ is of unit absolute value, then the induced diffeomorphism $\bar{\psi}$ of $\Gamma \backslash N$ is called a hyperbolic nilautomorphism, which is easily seen to be Anosov.

[^0]Such hyperbolic nilautomorphisms are just too many to be classified (see the recent paper [7]), which makes the following lemma meaningful.

Lemma 1. Let $\phi$ be a hyperbolic infranilautomorphism on an infranilmanifold $M$. If $\alpha$ is a $C^{\infty} 1$-form on $M$, such that $\mathrm{d} \alpha$ is $\phi$-invariant, then $\mathrm{d} \alpha=0$.

For a similar lemma in the case of the geodesic flows in negative curvature, see [5]. We deduce from Lemma 1 the following

Proposition 1. Let $\phi_{t}$ be the suspension of a hyperbolic infranilautomorphism. Then any smooth time change of $\phi_{t}$, which has also $C^{\infty}$ Anosov splitting, is homothetic to $\phi_{t}$.

Given two flows $\phi_{t}^{1}$ and $\phi_{t}^{2}$, such that the generator of $\phi_{t}^{1}$ is $X_{1}$. Then they are called homothetic, if $\exists a>0$, such that the flow of $a X_{1}$ is $C^{\infty}$ flow equivalent to $\phi_{t}^{2}$.

An Anosov flow is called geometric, if it preserves a smooth pseudo-Riemannian metric. Then we have

Proposition 2. Let $\phi_{t}$ be a geometric Anosov flow with smooth distributions on a closed manifold M, then the following two statements are equivalent,
(i) $\exists \gamma$, a closed $C^{\infty} 1$-form on $M$, such that $\gamma(X)>0$, where $X$ is the generator of $\phi_{t}$.
(ii) Up to a constant change of time scale, $\phi_{t}$ is $C^{\infty}$ flow equivalent to the suspension of a hyperbolic infranilautomorphism.

## 2. Proof of Lemma 1

As remarked above, each hyperbolic infranilautomorphism is normally and finitely covered by a hyperbolic nilautomorphism. So we need only prove Lemma 1 for hyperbolic nilautomorphisms. In the following, we denote by $\phi$ a hyperbolic nilautomorphism on a compact $n$-dimensional nilmanifold $\Gamma \backslash N$. Let $\alpha$ be a $C^{\infty} 1$-form on $\Gamma \backslash N$, such that $\mathrm{d} \alpha$ is $\phi$-invariant. To simplify the notation, we denote $\Gamma \backslash N$ also by $M$. Remark that any hyperbolic nilautomorphism is topologically transitive and has dense periodic orbits.

Consider the complexified vector bundles, $T M \otimes_{\mathbb{R}} \mathbb{C}$ and $T^{*} M \otimes_{\mathbb{R}} \mathbb{C}$. Any diffeomorphism of $M$ acts naturally on these two complex vector bundles and also on their exterior powers, by acting respectively on the real and imaginary parts. The brackets of the sections of $T M \otimes_{\mathbb{R}} \mathbb{C}$ are defined by extending $\mathbb{C}$-linearly the brackets of vector fields. Then for $\forall k \in \mathbb{N}$, the exterior derivation of the sections of $\wedge^{k}\left(T^{*} M \otimes_{\mathbb{R}} \mathbb{C}\right)$ is defined just as in the real case and is also denoted by d. For $\forall k \in \mathbb{Z}^{+}$, a section of $\wedge^{k}\left(T M \otimes_{\mathbb{R}} \mathbb{C}\right)$ or $\wedge^{k}\left(T^{*} M \otimes_{\mathbb{R}} \mathbb{C}\right)$ is called invariant, if its lift to $N$ is left-invariant. Any smooth $k$-form, $\gamma$, can be extended $\mathbb{C}$-linearly to a section of $\wedge^{k}\left(T^{*} M \otimes_{\mathbb{R}} \mathbb{C}\right)$, denoted by $\gamma^{\mathbb{C}}$. If $\gamma$ is $\phi$-invariant, then so is $\gamma^{\mathbb{C}}$. The idea of complexification is often useful, see for example [5].

Take independent invariant sections of $T M \otimes_{\mathbb{R}} \mathbb{C},\left\{X_{1}, \ldots, X_{n}\right\}$, such that the matrix $A$ of $\phi_{*}$ is in Jordan form:

$$
A=\left(\begin{array}{ccc}
A_{1} & & \\
& \ddots & \\
& & A_{k}
\end{array}\right) ; \quad A_{j}=\left(\begin{array}{cccc}
\lambda_{j} & 1 & & 0 \\
& \ddots & \ddots & \\
& & \ddots & 1 \\
0 & & & \lambda_{j}
\end{array}\right), \quad \forall 1 \leqslant j \leqslant k
$$

Then for $\forall m \geqslant n$, we have

$$
\left(A_{j}\right)^{m}=\left(\begin{array}{cccc}
\left(\lambda_{j}\right)^{m} & \binom{m}{1}\left(\lambda_{j}\right)^{m-1} & \ldots & \binom{m}{l-1}\left(\lambda_{j}\right)^{m-l+1} \\
0 & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \binom{m}{1}\left(\lambda_{j}\right)^{m-1} \\
0 & \ldots & 0 & \left(\lambda_{j}\right)^{m}
\end{array}\right), \quad \forall 1 \leqslant j \leqslant k
$$

where $l$ is the order of $A_{j}$. Denote by $\left\{X_{1}^{*}, \ldots, X_{n}^{*}\right\}$ the dual basis of $\left\{X_{1}, \ldots, X_{n}\right\}$. Remark that the matrix of $\phi^{*}$ is just $A^{T}$ in the basis $\left\{X_{1}^{*}, \ldots, X_{n}^{*}\right\}$. There exist complex functions $\left\{f_{i j}\right\}$ on $M$, such that

$$
(\mathrm{d} \alpha)^{\mathbb{C}}=\sum_{1 \leqslant i<j \leqslant n} f_{i j} X_{i}^{*} \wedge X_{j}^{*}
$$

Denote by $I_{j}$ the set of indices of $A_{j}$ in $A$. Take $i, j \in\{1, \ldots, k\}$ and define an order on $I_{i} \times I_{j}$ as follows: for $\forall(s, t),\left(s^{\prime}, t^{\prime}\right) \in I_{i} \times I_{j},(s, t)<\left(s^{\prime}, t^{\prime}\right)$ iff $s<s^{\prime}$ or if $s=s^{\prime}$, then $t<t^{\prime}$. Let $(k, l)$ be the smallest element of $I_{i} \times I_{j}$. For $\forall m \geqslant n$, we have $\left(\phi^{m}\right)^{*}(\mathrm{~d} \alpha)^{\mathbb{C}}=(\mathrm{d} \alpha)^{\mathbb{C}}$, i.e.,

$$
\begin{equation*}
(\mathrm{d} \alpha)^{\mathbb{C}}=\sum_{1 \leqslant i<j \leqslant n} f_{i j} \circ \phi^{m}\left(\phi^{*}\right)^{m} X_{i}^{*} \wedge\left(\phi^{*}\right)^{m} X_{j}^{*} \tag{*}
\end{equation*}
$$

By comparing the coefficients of $X_{k}^{*} \wedge X_{l}^{*}$ of both sides, we get

$$
f_{k l}=\left(\lambda_{i} \lambda_{j}\right)^{m}\left(f_{k l} \circ \phi^{m}\right)
$$

So if $\left|\lambda_{i} \lambda_{j}\right| \neq 1$, then $f_{k l} \equiv 0$. If $\left|\lambda_{i} \lambda_{j}\right|=1$ and $f_{k l} \not \equiv 0$, then by the density of the periodic orbits of $\phi, \exists x \in M$ and $p \in \mathbb{Z}^{+}$, such that $f_{k l}(x) \neq 0$ and $\phi^{p}(x)=x$. So $\left(\lambda_{i} \lambda_{j}\right)^{p}=1$. We deduce that $f_{k l} \circ\left(\phi^{p}\right)^{m}=f_{k l}$. But $\phi^{p}$ is also a hyperbolic nilautomorphism, so topologically transitive. Then $f_{k l}$ is constant.

Take $(s, t) \in I_{i} \times I_{j}$ and suppose that $f_{s^{\prime} t^{\prime}}$ is constant, for $\forall\left(s^{\prime}, t^{\prime}\right)<(s, t)$. Then by comparing the coefficients of $X_{s}^{*} \wedge X_{t}^{*}$ of both sides of $(*)$ and using the special form of $A^{m}$, we get

$$
f_{s t}=\left(\lambda_{i} \lambda_{j}\right)^{m}\left(f_{s t} \circ \phi^{m}+P(m)\right)
$$

where $P(m)$ is a polynomial without term of degree 0 . Then using the same arguments as above, we get that $f_{s t}$ is constant. So by induction on $I_{i} \times I_{j}, f_{a b}$ is seen to be constant, for $\forall(a, b) \in I_{i} \times I_{j}$. If in addition $\left|\lambda_{i} \lambda_{j}\right| \neq 1$, then we have in fact $f_{a b} \equiv 0$.

We deduce that $(\mathrm{d} \alpha)^{\mathbb{C}}$ is invariant. So $\mathrm{d} \alpha$ is also invariant on $M$. Then by [8], $\exists \beta$, an invariant 1 -form on $\Gamma \backslash N$, such that $\mathrm{d} \beta=\mathrm{d} \alpha$. Then we have $(\mathrm{d} \alpha)^{\mathbb{C}}=(\mathrm{d} \beta)^{\mathbb{C}}=\mathrm{d}\left(\beta^{\mathbb{C}}\right)$. Fix $i$, $j$, such that $\left|\lambda_{i} \lambda_{j}\right|=1$. Let $(k, l)$ be the smallest element of $I_{i} \times I_{j}$, then

$$
\phi_{*}\left[X_{k}, X_{l}\right]=\lambda_{i} \lambda_{j}\left[X_{k}, X_{l}\right] .
$$

Since $\phi$ is hyperbolic, then $\lambda_{i} \lambda_{j}$ can not be an eigenvalue of $\phi_{*}$. So $\left[X_{k}, X_{l}\right]=0$. Now by induction on $I_{i} \times I_{j}$, we get that $\left[X_{a}, X_{b}\right]=0$, for $\forall(a, b) \in I_{i} \times I_{j}$. Since $\beta^{\mathbb{C}}$ is invariant, then $f_{a b}=\mathrm{d}\left(\beta^{\mathbb{C}}\right)\left(X_{a}, X_{b}\right)$ $=-\beta^{\mathbb{C}}\left(\left[X_{a}, X_{b}\right]\right)=0$. We deduce that $(\mathrm{d} \alpha)^{\mathbb{C}}=0$. So $\mathrm{d} \alpha=0$.

## 3. Proof of Proposition 1

Let $\phi$ be a hyperbolic infranilautomorphism on an infranilmanifold $M$. Denote the suspension manifold by $\bar{M}$ and the suspension flow by $\phi_{t}^{X}$, where $X$ is the generator of the flow. Let $f X$ be a smooth time change of $\phi_{t}^{X}$ $(f>0)$, such that the flow of $f X, \phi_{t}^{f X}$, also has $C^{\infty}$ Anosov splitting. Denote the stable and unstable distributions of $\phi_{t}^{f X}$ by $E_{f X}^{-}$and $E_{f X}^{+}$. Let $\lambda_{1}$ be the canonical 1 -form of $f X$, i.e., $\lambda_{1}(f X)=1$ and $\lambda_{1}\left(E_{f X}^{ \pm}\right)=0$. Then $\mathrm{d} \lambda_{1}$ is
$\phi_{t}^{f X}$-invariant. So $i_{f X} \mathrm{~d} \lambda_{1}=0$. We deduce that $\mathrm{d} \lambda_{1}$ is $\phi_{t}$-invariant. So the restriction of $\mathrm{d} \lambda_{1}$ to the global section $M$ is $\phi$-invariant. Then by Lemma $1, \mathrm{~d} \lambda_{1}=0$. Let $\lambda$ be the canonical 1 -form of $\phi_{t}$, then $\mathrm{d} \lambda=0$. Since $H^{1}(\bar{M}, \mathbb{R}) \cong \mathbb{R}$ (see [9]), then $\exists a \in \mathbb{R}$, such that $\left[\lambda_{1}\right]=[a \cdot \lambda]$ in $H^{1}(\bar{M}, \mathbb{R})$. Then we get easily that $a>0$ and the flow of $\frac{X}{\lambda_{1}(X)}$ $(=f X)$ is $C^{\infty}$ flow equivalent to the flow of $\frac{1}{a} X$ (see [1] and [4]).

## 4. Proof of Proposition 2

If (ii) is true, then the canonical 1 -form fulfills (i). If (i) is true, then the flow of $\frac{X}{\gamma(X)}, \phi_{t}^{\gamma}$, is easily seen to be also a geometric Anosov flow with smooth distributions, of which the canonical 1 -form is $\gamma$ (see [2]). So $\phi_{t}^{\gamma}$ is $C^{\infty}$ flow equivalent to the suspension of a hyperbolic infranilautomorphism (see [2]). Let $\lambda$ be the canonical 1 -form of $\phi_{t}$. Then $\mathrm{d} \lambda$ is $\phi_{t}^{\gamma}$-invariant. Then by Lemma $1, \mathrm{~d} \lambda=0$. So (ii) is true.

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## References

[1] Y. Benoist, P. Foulon, F. Labourie, Flots d'Anosov à distributions stable et instable différentiables, J. Amer. Math. Soc. 5 (1992) 33-74.
[2] Y. Fang, Geometric Anosov flows of dimension 5 with smooth distributions, Preprint of I.R.M.A., No. 2003-009, Strasbourg.
[3] J. Franks, Anosov diffeomorphisms, in: Global Analysis, in: Proc. Symp. Pure Math., Vol. 14, 1970, pp. 61-93.
[4] É. Ghys, Flots d’Anosov dont les feuilletages stables sont différentiables, Ann. Sci. École Norm. Sup. (4) 20 (1987) 251-270.
[5] U. Hamenstadt, Invariant two-forms for geodesic flows, Math. Ann. 301 (4) (1995) 677-698.
[6] B. Hasselblatt, A. Katok, Introduction to the Modern Theory of Dynamical Systems, in: Encyclopedia of Mathematics and its Applications, Vol. 54, 1995.
[7] J. Lauret, Examples of Anosov diffeomorphisms, math.DS/0203126, 2002.
[8] M.S. Raghunathan, Discrete Subgroups of Lie Groups, Springer, Berlin, 1972.
[9] P. Tomter, Anosov flows on infra-homogeneous spaces, in: Global Analysis, in: Proc. Symp. Pure Math., Vol. XIV, 1970, pp. $299-327$.


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