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Dynamical Systems

A remark about hyperbolic infranilautomorphisms

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Abstract

We show that any exact 2-form, preserved by a hyperbolic infranilautomorphism, must be zero. We then deduce two propositions about geometric Anosov flows and the time change of suspensions. To cite this article: Y. Fang, C. R. Acad. Sci. Paris, Ser. I 336 (2003).

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Résumé

Une remarque sur les infranilautomorphismes hyperboliques. Nous montrons qu'une 2-forme exacte, préservée par un infranilautomorphisme hyperbolique, s'annule, et nous en déduisons deux propositions sur les flots d'Anosov géométriques et le changement du temps des suspensions. *Pour citer cet article : Y. Fang, C. R. Acad. Sci. Paris, Ser. I 336 (2003).* © 2003 Académie des sciences. Published by Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

1. Introduction

Let *M* be C^{∞} -closed manifold. A diffeomorphism ϕ of *M* is called Anosov, if there exists a ϕ -invariant splitting $TM = E^+ \oplus E^-$ and two positive numbers *a* and *b*, such that

$$\forall u^{\pm} \in E^{\pm}, \ \forall n \in \mathbb{Z}^+, \quad \left\| D\phi^{\mp n}(u^{\pm}) \right\| \leqslant c \, \mathrm{e}^{-bn} \| u^{\pm} \|,$$

where the distributions E^- and E^+ are called stable and unstable. If E^- and E^+ are C^∞ subbundles, then ϕ is said to have smooth distributions. Anosov flows and the corresponding notions are defined similarly (see [6]). The known examples of Anosov diffeomorphisms with smooth distributions are the hyperbolic infranilautomorphisms on some infranilmanifolds. Such diffeomorphisms are finitely covered by the hyperbolic nilautomorphisms, which are simply constructed as follows (see [3] for more details).

Let *N* be a connected and simply connected nilpotent Lie group and Γ be a lattice in *N*. Let ψ be an automorphism of *N* such that $\psi(\Gamma) = \Gamma$. If no eigenvalue of the differential $D_e \psi$ is of unit absolute value, then the induced diffeomorphism $\overline{\psi}$ of $\Gamma \setminus N$ is called a hyperbolic nilautomorphism, which is easily seen to be Anosov.

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Such hyperbolic nilautomorphisms are just too many to be classified (see the recent paper [7]), which makes the following lemma meaningful.

Lemma 1. Let ϕ be a hyperbolic infranilautomorphism on an infranilmanifold M. If α is a C^{∞} 1-form on M, such that $d\alpha$ is ϕ -invariant, then $d\alpha = 0$.

For a similar lemma in the case of the geodesic flows in negative curvature, see [5]. We deduce from Lemma 1 the following

Proposition 1. Let ϕ_t be the suspension of a hyperbolic infranilautomorphism. Then any smooth time change of ϕ_t , which has also C^{∞} Anosov splitting, is homothetic to ϕ_t .

Given two flows ϕ_t^1 and ϕ_t^2 , such that the generator of ϕ_t^1 is X_1 . Then they are called *homothetic*, if $\exists a > 0$, such that the flow of aX_1 is C^{∞} flow equivalent to ϕ_t^2 .

An Anosov flow is called geometric, if it preserves a smooth pseudo-Riemannian metric. Then we have

Proposition 2. Let ϕ_t be a geometric Anosov flow with smooth distributions on a closed manifold M, then the following two statements are equivalent,

- (i) $\exists \gamma$, a closed C^{∞} 1-form on M, such that $\gamma(X) > 0$, where X is the generator of ϕ_t .
- (ii) Up to a constant change of time scale, ϕ_t is C^{∞} flow equivalent to the suspension of a hyperbolic infranilautomorphism.

2. Proof of Lemma 1

As remarked above, each hyperbolic infranilautomorphism is normally and finitely covered by a hyperbolic nilautomorphism. So we need only prove Lemma 1 for hyperbolic nilautomorphisms. In the following, we denote by ϕ a hyperbolic nilautomorphism on a compact *n*-dimensional nilmanifold $\Gamma \setminus N$. Let α be a C^{∞} 1-form on $\Gamma \setminus N$, such that $d\alpha$ is ϕ -invariant. To simplify the notation, we denote $\Gamma \setminus N$ also by M. Remark that any hyperbolic nilautomorphism is topologically transitive and has dense periodic orbits.

Consider the complexified vector bundles, $TM \otimes_{\mathbb{R}} \mathbb{C}$ and $T^*M \otimes_{\mathbb{R}} \mathbb{C}$. Any diffeomorphism of M acts naturally on these two complex vector bundles and also on their exterior powers, by acting respectively on the real and imaginary parts. The brackets of the sections of $TM \otimes_{\mathbb{R}} \mathbb{C}$ are defined by extending \mathbb{C} -linearly the brackets of vector fields. Then for $\forall k \in \mathbb{N}$, the exterior derivation of the sections of $\wedge^k(T^*M \otimes_{\mathbb{R}} \mathbb{C})$ is defined just as in the real case and is also denoted by d. For $\forall k \in \mathbb{Z}^+$, a section of $\wedge^k(TM \otimes_{\mathbb{R}} \mathbb{C})$ or $\wedge^k(T^*M \otimes_{\mathbb{R}} \mathbb{C})$ is called invariant, if its lift to N is left-invariant. Any smooth k-form, γ , can be extended \mathbb{C} -linearly to a section of $\wedge^k(T^*M \otimes_{\mathbb{R}} \mathbb{C})$, denoted by $\gamma^{\mathbb{C}}$. If γ is ϕ -invariant, then so is $\gamma^{\mathbb{C}}$. The idea of complexification is often useful, see for example [5].

Take independent invariant sections of $TM \otimes_{\mathbb{R}} \mathbb{C}$, $\{X_1, \ldots, X_n\}$, such that the matrix A of ϕ_* is in Jordan form:

$$A = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{pmatrix}; \quad A_j = \begin{pmatrix} \lambda_j & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_j \end{pmatrix}, \quad \forall 1 \leq j \leq k.$$

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Then for $\forall m \ge n$, we have

$$(A_{j})^{m} = \begin{pmatrix} (\lambda_{j})^{m} & \binom{m}{1} (\lambda_{j})^{m-1} & \dots & \binom{m}{l-1} (\lambda_{j})^{m-l+1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \binom{m}{1} (\lambda_{j})^{m-1} \\ 0 & \dots & 0 & (\lambda_{j})^{m} \end{pmatrix}, \quad \forall 1 \leq j \leq k,$$

where *l* is the order of A_j . Denote by $\{X_1^*, \ldots, X_n^*\}$ the dual basis of $\{X_1, \ldots, X_n\}$. Remark that the matrix of ϕ^* is just A^T in the basis $\{X_1^*, \ldots, X_n^*\}$. There exist complex functions $\{f_{ij}\}$ on *M*, such that

$$(\mathrm{d}\alpha)^{\mathbb{C}} = \sum_{1 \leqslant i < j \leqslant n} f_{ij} X_i^* \wedge X_j^*.$$

Denote by I_j the set of indices of A_j in A. Take $i, j \in \{1, ..., k\}$ and define an order on $I_i \times I_j$ as follows: for $\forall (s, t), (s', t') \in I_i \times I_j, (s, t) < (s', t')$ iff s < s' or if s = s', then t < t'. Let (k, l) be the smallest element of $I_i \times I_j$. For $\forall m \ge n$, we have $(\phi^m)^* (d\alpha)^{\mathbb{C}} = (d\alpha)^{\mathbb{C}}$, i.e.,

$$(\mathrm{d}\alpha)^{\mathbb{C}} = \sum_{1 \leqslant i < j \leqslant n} f_{ij} \circ \phi^m \ (\phi^*)^m X_i^* \wedge (\phi^*)^m X_j^*. \tag{*}$$

By comparing the coefficients of $X_k^* \wedge X_l^*$ of both sides, we get

$$f_{kl} = (\lambda_i \lambda_j)^m (f_{kl} \circ \phi^m).$$

So if $|\lambda_i \lambda_j| \neq 1$, then $f_{kl} \equiv 0$. If $|\lambda_i \lambda_j| = 1$ and $f_{kl} \neq 0$, then by the density of the periodic orbits of ϕ , $\exists x \in M$ and $p \in \mathbb{Z}^+$, such that $f_{kl}(x) \neq 0$ and $\phi^p(x) = x$. So $(\lambda_i \lambda_j)^p = 1$. We deduce that $f_{kl} \circ (\phi^p)^m = f_{kl}$. But ϕ^p is also a hyperbolic nilautomorphism, so topologically transitive. Then f_{kl} is constant.

Take $(s, t) \in I_i \times I_j$ and suppose that $f_{s't'}$ is constant, for $\forall (s', t') < (s, t)$. Then by comparing the coefficients of $X_s^* \wedge X_t^*$ of both sides of (*) and using the special form of A^m , we get

$$f_{st} = (\lambda_i \lambda_i)^m (f_{st} \circ \phi^m + P(m)),$$

where P(m) is a polynomial without term of degree 0. Then using the same arguments as above, we get that f_{st} is constant. So by induction on $I_i \times I_j$, f_{ab} is seen to be constant, for $\forall (a, b) \in I_i \times I_j$. If in addition $|\lambda_i \lambda_j| \neq 1$, then we have in fact $f_{ab} \equiv 0$.

We deduce that $(d\alpha)^{\mathbb{C}}$ is invariant. So $d\alpha$ is also invariant on M. Then by [8], $\exists \beta$, an invariant 1-form on $\Gamma \setminus N$, such that $d\beta = d\alpha$. Then we have $(d\alpha)^{\mathbb{C}} = (d\beta)^{\mathbb{C}} = d(\beta^{\mathbb{C}})$. Fix i, j, such that $|\lambda_i \lambda_j| = 1$. Let (k, l) be the smallest element of $I_i \times I_j$, then

$$\phi_*[X_k, X_l] = \lambda_i \lambda_j [X_k, X_l].$$

Since ϕ is hyperbolic, then $\lambda_i \lambda_j$ can not be an eigenvalue of ϕ_* . So $[X_k, X_l] = 0$. Now by induction on $I_i \times I_j$, we get that $[X_a, X_b] = 0$, for $\forall (a, b) \in I_i \times I_j$. Since $\beta^{\mathbb{C}}$ is invariant, then $f_{ab} = d(\beta^{\mathbb{C}})(X_a, X_b) = -\beta^{\mathbb{C}}([X_a, X_b]) = 0$. We deduce that $(d\alpha)^{\mathbb{C}} = 0$. So $d\alpha = 0$.

3. Proof of Proposition 1

Let ϕ be a hyperbolic infranilautomorphism on an infranilmanifold M. Denote the suspension manifold by \overline{M} and the suspension flow by ϕ_t^X , where X is the generator of the flow. Let fX be a smooth time change of ϕ_t^X (f > 0), such that the flow of fX, ϕ_t^{fX} , also has C^{∞} Anosov splitting. Denote the stable and unstable distributions of ϕ_t^{fX} by E_{fX}^- and E_{fX}^+ . Let λ_1 be the *canonical* 1-form of fX, i.e., $\lambda_1(fX) = 1$ and $\lambda_1(E_{fX}^{\pm}) = 0$. Then $d\lambda_1$ is

 ϕ_t^{fX} -invariant. So $i_{fX}d\lambda_1 = 0$. We deduce that $d\lambda_1$ is ϕ_t -invariant. So the restriction of $d\lambda_1$ to the global section M is ϕ -invariant. Then by Lemma 1, $d\lambda_1 = 0$. Let λ be the *canonical* 1-*form* of ϕ_t , then $d\lambda = 0$. Since $H^1(\overline{M}, \mathbb{R}) \cong \mathbb{R}$ (see [9]), then $\exists a \in \mathbb{R}$, such that $[\lambda_1] = [a \cdot \lambda]$ in $H^1(\overline{M}, \mathbb{R})$. Then we get easily that a > 0 and the flow of $\frac{X}{\lambda_1(X)}$ (= fX) is C^{∞} flow equivalent to the flow of $\frac{1}{a}X$ (see [1] and [4]).

4. Proof of Proposition 2

If (ii) is true, then the *canonical* 1-*form* fulfills (i). If (i) is true, then the flow of $\frac{X}{\gamma(X)}$, ϕ_t^{γ} , is easily seen to be also a geometric Anosov flow with smooth distributions, of which the *canonical* 1-*form* is γ (see [2]). So ϕ_t^{γ} is C^{∞} flow equivalent to the suspension of a hyperbolic infranilautomorphism (see [2]). Let λ be the *canonical* 1-*form* of ϕ_t . Then d λ is ϕ_t^{γ} -invariant. Then by Lemma 1, d $\lambda = 0$. So (ii) is true.

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