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Ordinary Differential Equations

On the C^1 normal forms for hyperbolic vector fields ☆

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Abstract

Given two germs of hyperbolic vector fields associated to autonomous ordinary differential equations $\dot{x} = Ax + \cdots$ and $\dot{y} = By + \cdots$, where $x, y \in \mathbf{R}^n$, and A and B are $n \times n$ matrices, we prove that under some algebraic conditions on the eigenvalues of the matrices and genericity condition on the nonlinear terms, they are at least C^1 conjugate if and only if A and B are similar. ***To cite this article: Z. Ren, J. Yang, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Sur la classification C^1 des champs de vecteurs hyperboliques. Etant donnés deux germes de champs de vecteurs hyperboliques définis par des équations différentielles autonomes $\dot{x} = Ax + \cdots$ et $\dot{y} = By + \cdots$, où $x, y \in \mathbf{R}^n$, A et B sont des matrices d'ordre n, on démontre que, sous certaines conditions algébriques sur les valeurs propres des matrices et des conditions de non dégénérescensce des terms nonlinéaires, ils sont au moins C^1 conjugués si et seulement si A et B sont semblables. ***Pour citer cet article : Z. Ren, J. Yang, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

1. Introduction and main results

Denote by $V_A(n)$ the set of germs at the origin of hyperbolic vector fields $X_A(n)$ associated to autonomous ordinary differential equations

$$\dot{x} = Ax + \cdots, \quad x \in \mathbf{R}^n, \tag{1}$$

where A is the differential of $X_A(n)$ at the origin. The present paper is devoted to studying the C^1 classification of a generic vector field $X_A(n) \in V_A(n)$.

Throughout the paper, we shall assume that the vector field (1) is C^∞, although in many cases, the results also hold for vector fields having only finite degree of smoothness.

The main results of the present paper are the following theorems.

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Theorem 1.1. *Let vector field $X_A(n)$ be associated to the differential equation* (1) *which is assumed to have a generic* 2*-jet. Then*

(i) *for $n \leqslant 4$, the matrix A entirely determines the C^1 classification of $X_A(n)$. Namely, two such vector fields $X_{A_1}(n)$ and $X_{A_2}(n)$ are C^1 conjugate* (*orbitally equivalent, resp.*) *if and only if A_1 and A_2 are strictly similar* (*similar, resp.*).

(ii) *for $n = 5$, the statement* (i) *holds, too, with the following exceptions*: *the eigenvalues of A, up to ennumeration, have the forms*

$$(-\alpha, -\alpha \pm \beta, \pm\alpha + \beta, \beta, \beta), \quad \text{or} \quad (-\alpha, -\alpha, -\alpha + \beta, \beta, \beta), \tag{2}$$

where $\alpha\beta > 0$, while A itself either it is diagonalizable or it has two 2*-dimensional Jordan blocks.*

Here we call two matrices A and B *strictly similar* (resp. *similar*) if there is an invertible matrix T and a non-zero real number μ such that $A = TBT^{-1}$ (resp., $A = \mu TBT^{-1}$).

We remark that the two pairs of signs $\pm$ in (2) are independent. Thus (2) in fact contains five cases. From the above theorems, we immediately have the following.

Corollary 1.2. *For $n \leqslant 5$, if all the eigenvalues of A are mutually distinct and the* 2*-jet of $X_A(n)$ is generic, then the matrix A entirely determines the C^1 classification of the whole vector field $X_A(n)$.*

Example 1. Let A be a 5×5 matrix with eigenvalues $(-\alpha, -\alpha, -\alpha + \beta, \beta, \beta)$ but with only one 2-dimensional Jordan block (a non-exceptional case in (2)). Then $X_A(5)$, although with two possible positions of the Jordan block, up to ennumeration of coordinates, is C^1 conjugate to the moduli-free normal form

$$\dot{x}_1 = -\alpha x_1 + x_2, \quad \dot{x}_2 = -\alpha x_2, \quad \dot{x}_3 = (-\alpha + \beta)x_3 + x_1x_4 + x_1x_5, \quad \dot{x}_4 = \beta x_4, \quad \dot{x}_5 = \beta x_5.$$

The genericity conditions, in terms of (independent of, though) the final coordinates, ask the non-vanishing of the coefficients of $x_1x_4\partial/\partial x_3$ and $x_1x_5\partial/\partial x_3$.

To state the general result on $\mathbf{R}^n$, denote the eigenvalues of A by

$$\lambda = (\mu_1, \ldots, \mu_s, \nu_1, \ldots, \nu_u), \quad \operatorname{Re}\mu_i < 0, \ \operatorname{Re}\nu_j > 0, \ s + u = n, \tag{3}$$

and we have

Theorem 1.3. *For vector field $X_A(n)$, if all the eigenvalues of A are mutually distinct and satisfy the inequalities*

$$\operatorname{Re}(\mu_{i_1} - \mu_{i_2}) \neq \operatorname{Re}(\nu_{j_1} - \nu_{j_2}), \quad i_1 \neq i_2, \ j_1 \neq j_2, \tag{4}$$

and the 2*-jet of $X_A(n)$ is generic, then the matrix A entirely determines the C^1 classification of $X_A(n)$.*

Alternatively, we present another set of sufficient conditions which lead to the C^1 linearization of $X_A(n)$ (consequently its C^1 classification, regardless the genericity or degeneracy of the nonlinear part). Recall the results of [1] on the C^1 linearization: *if the eigenvalues of the matrix A satisfy the inequalities* $\operatorname{Re}\lambda_i \neq \operatorname{Re}\lambda_k + \operatorname{Re}\lambda_l$, *where $\lambda_{i,k,l}$ run over all the eigenvalues of the A, then $X_A(n)$ can be C^1 linearized.*

Theorem 1.4. *If the eigenvalues of A satisfy the following inequalities*

$$\operatorname{Re}\mu_i \neq \operatorname{Re}\mu_{j_1} + \operatorname{Re}\nu_{j_2}, \quad \textit{and} \quad \operatorname{Re}\nu_i \neq \operatorname{Re}\mu_{j_1} + \operatorname{Re}\nu_{j_2}, \tag{5}$$

then $X_A(n)$ is C^1 linearizable.

We remark that conditions (4) and (5) are mutually independent. We remark also that the number of inequalities (5) can be essentially less than that of [1] since the right sides of (5) take pairs of eigenvalues with one negative real part and one positive real part.

2. Preliminaries and sketch of the proofs

It is well known, according to Sternberg [7,8] and Chen [5], that to study finitely smooth normal forms of C^∞ vector fields at a hyperbolic equilibrium point, it suffices to consider truncated polynomial vector fields whose nonlinear part consists of resonant monomials only. In nilpotent cases, based on the linear part of the vector field, one can further apply the Belitskii theorem [1] to delete more terms from the polynomial. We collect these facts into the following

Proposition 2.1. *In a neighborhood of a hyperbolic equilibrium point, an infinitely smooth vector field can be reduced, by a C^r $(r \geqslant 0)$ change of coordinates, to a polynomial resonant normal form $\dot{x} = Jx + P(x)$, where J is an $n \times n$ matrix in its Jordan form, P is a polynomial satisfying $J^t P(x) - P'(x) J^t x = 0$, and the order of P depends on r.*

We refer the reader to [4] for an extensive exposition on the relation between the order of P, the number r, and the smoothness of the vector fields.

Another important theorem we shall use in the proof of the theorem is the Samovol theorem on the linearization of hyperbolic vector fields.

Proposition 2.2 (Samovol [6]). *Let X be a smooth hyperbolic vector field whose eigenvalues $(\mu_1, \ldots, \mu_s, \nu_1, \ldots, \nu_u)$ are ordered as*

$$\operatorname{Re}\mu_s \leqslant \cdots \leqslant \operatorname{Re}\mu_1 < 0 < \operatorname{Re}\nu_1 \leqslant \cdots \leqslant \operatorname{Re}\nu_u.$$

Suppose that the following $S(k)$ condition holds for some positive integer k: for each resonant relation of the form

$$\mu_j = \sum r_m^- \mu_m + \sum r_h^+ \nu_h, \quad \text{or} \quad \nu_j = \sum r_m^- \mu_m + \sum r_h^+ \nu_h \tag{6}$$

there is an $m \leqslant s$ or an $h \leqslant u$ such that

$$k \operatorname{Re}\nu_h < \operatorname{Re}\big(r_1^+ \nu_1 + \cdots + r_h^+ \nu_h\big), \quad \text{or} \quad k \operatorname{Re}\mu_m > \operatorname{Re}\big(r_1^- \mu_1 + \cdots + r_m^- \mu_m\big).$$

Then X is C^k conjugate to its linear part $j^1 X$.

In the proof of our results, we in fact use the Samovol theorem in the following way. If a resonant monomial of X satisfies the mentioned $S(k)$ condition, then the monomial can be eliminated by a C^k change of coordinates. In other words, instead of linearizing the vector field X, following the normalization techniques developed in [2], we can show that X is C^k equivalent to $\widetilde{X}$ which is obtained from X simply by eliminating the resonant monomials satisfying $S(k)$ conditions. Thus the elimination of resonant monomials can be done term by term, leaving unchanged those resonant monomials which do not meet $S(k)$ condition. For example, the elimination of two resonant monomials in the vector field $(-x_1, -2x_2, x_3, 2x_4, 3x_5 + x_1 x_2 x_4^3 + x_1^3 x_4^3)$ can be fulfilled respectively via the normalizations

$$x_j = y_j, \quad j = 1, 2, 3, 4, \quad x_5 = y_5 - \frac{1}{6} y_1 y_2 y_4^3 \ln\big(|y_1|^6 + |y_2|^3\big)$$

and

$$y_j = z_j, \quad j = 1, 2, 3, 4, \quad y_5 = z_5 - \frac{1}{4} z_1^3 z_4^3 \ln\big(|z_1|^4 + |z_2|^2\big).$$

The proof to each theorem consists of two main steps: first, we show that, under the conditions stated in the theorems, the 2-jet of a given germ of a vector field can be normalized to a moduli-free normal form. More precisely, by applying the Poincaré–Dulac–Belitskii theorem we can normalize the 2-jet of the vector field and scale all the coefficients of the quadratic terms to 1. To show that the final normal form is the simplest or is independent of an individual coordinate system, it suffices to show that under resonant changes of coordinates (see [3]), the normal form is invariant (in certain cases, we can also see this point from geometry, i.e., by counting the number of invariant surfaces of the 2-jet of the vector field). Then, essentially following the Samavol theorem, we show the following: any vector field, under the theorem conditions, is 2-jet determined with respect to C^1 classification. In other words, we shall show that any nonlinear terms with order equal to or higher than three can be eliminated by a C^1 change of coordinates.

In particular, in lower dimensional cases, namely, the cases listed in Theorem 1.1, the proof to the theorem can be done by exhausting all possible algebraic structures of the eigenvalues. Thus some straightforward calculation convinces us that no restriction on the eigenvalues is needed.

In the general case as discussed in Theorem 1.3, under the set of algebraic conditions, we in fact can fulfil the above normalization procedures (particularly, the linear scalings), observing the following facts: (i) there is at most one resonant monomial of the form $x_i y_j$ attached to each component differential equation $\dot{x}_h$ or $\dot{y}_k$, where x_i and y_j represent stable and unstable coordinates, respectively; (ii) each coordinate x_i (resp. y_j) can occur at most once in $x_i y_j \partial/\partial x$ (resp. $x_i y_j \partial/\partial y$). Namely, if there are two resonant monomials $x_{i_1} y_{j_1} \partial/\partial x_{h_1}$ and $x_{i_2} y_{j_2} \partial/\partial x_{h_2}$, then $i_1 \neq i_2$ (resp. if there are two resonant monomials $x_{i_1} y_{j_1} \partial/\partial y_{k_1}$ and $x_{i_2} y_{j_2} \partial/\partial y_{k_2}$, then $j_1 \neq j_2$).

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