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Probability Theory

A type of time-symmetric forward–backward stochastic differential equations

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Abstract

In this Note, we study a type of time-symmetric forward–backward stochastic differential equations. Under some monotonicity assumptions, we establish the existence and uniqueness theorem by means of a method of continuation. We also give an application. ***To cite this article: S. Peng, Y. Shi, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Un type d'équations différentielles stochastiques progressives–rétrogrades symétriques par rapport au temps. Nous étudions dans cette Note un type d'équations différentielles stochastiques progressives–rétrogrades symétriques par rapport au temps. Sous certaines conditions de monotonie, nous donnons un théorème d'existence et unicité des solutions des équations par une méthode de continuation. Ensuite nous présentons une application. ***Pour citer cet article : S. Peng, Y. Shi, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

Dans cette Note, nous étudions un type d'équations différentielles stochastiques progressives–rétrogrades symétriques par rapport au temps. Précisément, nous considérons les équations suivantes :

$$\begin{cases} \mathrm{d}y_t = f(t, y_t, Y_t, z_t, Z_t)\,\mathrm{d}t + g(t, y_t, Y_t, z_t, Z_t)\,\mathrm{d}W_t - z_t\,\mathrm{d}B_t, & y_0 = x, \\ \mathrm{d}Y_t = F(t, y_t, Y_t, z_t, Z_t)\,\mathrm{d}t + G(t, y_t, Y_t, z_t, Z_t)\,\mathrm{d}B_t + Z_t\,\mathrm{d}W_t, & Y_T = \Phi(y_T), \end{cases} \tag{1}$$

où $\{W_t\}_{0\leqslant t}$ et $\{B_t\}_{0\leqslant t}$ sont deux mouvements browniens indépendants, et $(\mathrm{d}W_t)$ est une integrale de Itô (standard) progressive, tandis que $(\mathrm{d}B_t)$ est une integrale de Itô rétrograde commencée en T, et inverse en temps.

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Cela généralise les types d'équations différentielles stochastiques progressive–rétrogrades qui ont été auparavant étudiés. Par une méthode de continuation, on a le résultat suivant :

Théorème. *On suppose les hypothèses* (H1)–(H4) *satisfaites. Alors, pour chaque* $x \in \mathbb{R}^n$*, l'Éq.* (1) *a une solution unique dans* $M^2(0, T; \mathbb{R}^{n+n+n\times l+n\times d})$.

1. Introduction

Since a series of research by Antonelli [1] and specially by Ma, Protter and Yong [6], with applications in finance, forward–backward stochastic differential equations (FBSDE in short) have been deeply investigated (see [4]). One of these directions was initialized by Hu and Peng [5] and developed by Peng and Wu [12], Yong [13], Peng and Shi [11] and Peng [10], which generalized stochastic Hamiltonian systems introduced by Bismut [3] in 1973 and then systematically investigated by Bensoussan [2]. In general, a FBSDE consists of a forward SDE of Itô's type, and a backward SDE of Pardoux–Peng's type (see [7]). They are coupled with each other. However, this type of FBSDE is not symmetric with respect to time. In this paper we will study a type of time-symmetric FBSDE, i.e., the forward equation is "forward" with respect to a standard stochastic integral $\mathrm{d}W_t$, as well as "backward" with respect to a backward stochastic integral $\mathrm{d}B_t$; the coupled "backward equation" is "forward" under the backward stochastic integral $\mathrm{d}B_t$ and "backward" under the forward one. In other words, both the forward equation and the backward one are types of SDEs introduced by Pardoux and Peng [8] under the name "backward doubly SDE" with different directions of stochastic integral. We will also discuss the corresponding time-symmetric stochastic Hamiltonian systems.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $[0, T]$ be a fixed arbitrarily large time duration throughout this paper. Let $\{W_t;\ 0 \leqslant t \leqslant T\}$ and $\{B_t;\ 0 \leqslant t \leqslant T\}$ be two mutually independent standard Brownian motions defined on $(\Omega, \mathcal{F}, P)$, with values respectively in $\mathbb{R}^d$ and in $\mathbb{R}^l$. Let $\mathcal{N}$ denote the class of P-null elements of $\mathcal{F}$. For each $t \in [0, T]$, we define $\mathcal{F}_t \doteq \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B$, where $\mathcal{F}_t^W = \mathcal{N} \vee \sigma\{W_r - W_0;\ 0 \leqslant r \leqslant t\}$, $\mathcal{F}_{t,T}^B = \mathcal{N} \vee \sigma\{B_r - B_t;\ t \leqslant r \leqslant T\}$. Note that the collection $\{\mathcal{F}_t,\ t \in [0, T]\}$ is neither increasing nor decreasing, and it does not constitute a filtration. Let $M^2(0, T; \mathbb{R}^n)$ denote the space of all (classes of $\mathrm{d}P \times \mathrm{d}t$ a.e. equal) $\mathbb{R}^n$-valued $\mathcal{F}_t$-measurable stochastic processes $\{v_t;\ t \in [0, T]\}$ which satisfy $\mathbf{E}\int_0^T |v_t|^2\,\mathrm{d}t < \infty$. Obviously $M^2(0, T; \mathbb{R}^n)$ is a Hilbert space. For a given $u \in M^2(0, T; \mathbb{R}^d)$ and $v \in M^2(0, T; \mathbb{R}^l)$, one can define the (standard) forward Itô's integral $\int_0^{\cdot} u_s\,\mathrm{d}W_s$ and the backward Itô's integral $\int_{\cdot}^T v_s\,\mathrm{d}B_s$. They are both in $M^2(0, T; \mathbb{R})$. (See [8] for details.)

Under this framework, we consider the following type of time-symmetric forward–backward stochastic differential equations (SFBSDE in short)

$$\begin{cases} y_t = x + \displaystyle\int_0^t f(s, y_s, Y_s, z_s, Z_s)\,\mathrm{d}s + \int_0^t g(s, y_s, Y_s, z_s, Z_s)\,\mathrm{d}W_s - \int_0^t z_s\,\mathrm{d}B_s, \\ Y_t = \Phi(y_T) + \displaystyle\int_t^T F(s, y_s, Y_s, z_s, Z_s)\,\mathrm{d}s + \int_t^T G(s, y_s, Y_s, z_s, Z_s)\,\mathrm{d}B_s + \int_t^T Z_s\,\mathrm{d}W_s. \end{cases} \tag{1}$$

In the case when (1) does not involve the term of backward Itô's integral, that is, when $G \equiv 0$ and $f,\ g,\ F$ are independent of z, this system will degenerate to the FBSDE which has been studied by Hu and Peng [5] and so on. On the other hand, a new kind of backward stochastic differential equations, called backward doubly stochastic differential equations, has been introduced by Pardoux and Peng [8]. The aim of this Note is to combine the above two types of results, to study the existence and uniqueness of a solution to (1). Under some monotonicity conditions (see (H1) and (H2)), we will apply the method of continuation to solve (1). This method was introduced by Peng in [9] for solving backward stochastic differential equations (BSDE in short) with random terminal time and then in [5] and [12] and [13] for solving FBSDE.

It is an interesting open problem to learn how to connect this type of SFBSDE to some nonlinear stochastic partial differential equations in order to generalize the well-known nonlinear Feynman–Kac formula to the stochastic case. In the interest of studying stochastic viscosity solutions for nonlinear stochastic PDEs by means of this type of SFBSDE, it is indispensable to some types of comparison theorems of SFBSDE. It is worth noting that the comparison theorem of SFBSDE is interesting in its own right, as it seems not to naturally derive from the one of FBSDE; as a result the collection $\{\mathcal{F}_t\}_{t\in[0,T]}$ is not a natural filtration. We hope to be able to address this issue in our future publications.

This Note is organized as follows: in the next section we present our main results; in Section 3 we provide some a priori estimates; the estimates will be applied to prove the existence and uniqueness theorem in Section 4; finally in Section 5 we will apply the above result to a doubly stochastic Hamiltonian system.

For the simplicity of notations, we only consider the case where y and Y take the same dimension. But using the techniques introduced by Peng and Wu [12], we can also treat some more general cases.

2. Setting of the problem and the main results

Consider the following type of time-symmetric forward–backward stochastic differential equations

$$\begin{cases} \mathrm{d}y_t = f(t, y_t, Y_t, z_t, Z_t)\,\mathrm{d}t + g(t, y_t, Y_t, z_t, Z_t)\,\mathrm{d}W_t - z_t\,\mathrm{d}B_t, & y_0 = x, \\ \mathrm{d}Y_t = F(t, y_t, Y_t, z_t, Z_t)\,\mathrm{d}t + G(t, y_t, Y_t, z_t, Z_t)\,\mathrm{d}B_t + Z_t\,\mathrm{d}W_t, & Y_T = \Phi(y_T), \end{cases} \tag{2}$$

where $x \in \mathbb{R}^n$,

$$\begin{aligned}
&F : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^{n\times d} \to \mathbb{R}^n, \\
&f : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^{n\times d} \to \mathbb{R}^n, \\
&G : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^{n\times d} \to \mathbb{R}^{n\times l}, \\
&g : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^{n\times d} \to \mathbb{R}^{n\times d}, \\
&\Phi : \Omega \times \mathbb{R}^n \to \mathbb{R}^n.
\end{aligned}$$

Let us introduce some notations $\zeta = (y, Y, z, Z)$, $A(t,\zeta) = (F, f, G, g)(t,\zeta)$. We use the usual inner product $\langle\cdot,\cdot\rangle$ and Euclidean norm $|\cdot|$ in $\mathbb{R}^n$, $\mathbb{R}^{n\times l}$ and $\mathbb{R}^{n\times d}$. All the equalities and inequalities mentioned in this paper are in the sense of $\mathrm{d}t \times \mathrm{d}P$ almost surely on $[0,T]\times\Omega$.

Definition 2.1. A quadruple of $\mathcal{F}_t$-measurable stochastic processes $(y, Y, z, Z) \in M^2\big(0,T;\mathbb{R}^{n+n+n\times l+n\times d}\big)$ is called a solution of SFBSDE (2), if (2) is satisfied.

The following monotonicity conditions are our main assumptions:

(H1) There exists a constant $\mu > 0$, such that

$$\begin{aligned}
&\big\langle A(t,\zeta) - A(t,\bar\zeta), \zeta - \bar\zeta\big\rangle \leqslant -\mu|\zeta - \bar\zeta|^2, \\
&\quad \forall \zeta = (y, Y, z, Z),\ \bar\zeta = \big(\bar y, \overline{Y}, \bar z, \overline{Z}\big) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times l} \times \mathbb{R}^{n\times d},\ \forall t \in [0,T].
\end{aligned}$$

(H2) $\langle \Phi(y) - \Phi(\bar y), y - \bar y\rangle \geqslant 0$, $\forall y, \bar y \in \mathbb{R}^n$.

We also assume that

(H3) For each $\zeta \in \mathbb{R}^{n+n+n\times l+n\times d}$, $A(\cdot,\zeta)$ is a $\mathcal{F}_t$-measurable vector process defined on $[0,T]$ with $A(\cdot,0) \in M^2(0,T;\mathbb{R}^{n+n+n\times l+n\times d})$, and for each $y \in \mathbb{R}^n$, $\Phi(y)$ is a $\mathcal{F}_T$-measurable random vector with $\Phi(0) \in L^2(\Omega,\mathcal{F}_T,P;\mathbb{R}^n)$.

(H4) $A(t,\zeta)$ and $\Phi(y)$ satisfy Lipschitz condition: there exists a constant $k>0$, such that

$$\big|A(t,\zeta)-A(t,\bar\zeta)\big|\leqslant k|\zeta-\bar\zeta|,\quad \forall\zeta,\bar\zeta\in\mathbb{R}^{n+n+n\times l+n\times d},\ \forall t\in[0,T],$$
$$\big|\Phi(y)-\Phi(\bar y)\big|\leqslant k|y-\bar y|,\quad \forall y,\bar y\in\mathbb{R}^n.$$

Our main result is as follows.

Theorem 2.2. *Under assumptions* (H1)–(H4)*, for each* $x\in\mathbb{R}^n$*,* (2) *has a unique solution in* $M^2(0,T;\mathbb{R}^{n+n+n\times l+n\times d})$.

3. A priori estimates

In order to prove the existence and uniqueness result for (2), we need the following lemmas. They involve a priori estimates of solutions of the following family of SFBSDEs parametrized by $\alpha\in[0,1]$.

$$\begin{cases}\mathrm{d}y_t=\big[f^\alpha(t,U_t)+f_0(t)\big]\,\mathrm{d}t-z_t\,\mathrm{d}B_t+\big[g^\alpha(t,U_t)+g_0(t)\big]\,\mathrm{d}W_t, & y_0=x,\\ \mathrm{d}Y_t=\big[F^\alpha(t,U_t)+F_0(t)\big]\,\mathrm{d}t+Z_t\,\mathrm{d}W_t+\big[G^\alpha(t,U_t)+G_0(t)\big]\,\mathrm{d}B_t, & Y_T=\Phi^\alpha(y_T)+\varphi,\end{cases}\tag{3}$$

where $U=(y,Y,z,Z)$ and for any given $\alpha\in[0,1]$,

$$f^\alpha(t,y,Y,z,Z)=\alpha f(t,y,Y,z,Z)-(1-\alpha)Y,\qquad g^\alpha(t,y,Y,z,Z)=\alpha g(t,y,Y,z,Z)-(1-\alpha)Z,$$
$$F^\alpha(t,y,Y,z,Z)=\alpha F(t,y,Y,z,Z)-(1-\alpha)y,\qquad G^\alpha(t,y,Y,z,Z)=\alpha G(t,y,Y,z,Z)-(1-\alpha)z,$$
$$\Phi^\alpha(y)=\alpha\Phi(y)+(1-\alpha)y.$$

Observe that when $\alpha=0$, (3) is written in the following simple form

$$\begin{cases}\mathrm{d}y_t=\big(-Y_t+f_0(t)\big)\,\mathrm{d}t+\big(-Z_t+g_0(t)\big)\,\mathrm{d}W_t-z_t\,\mathrm{d}B_t, & y_0=x,\\ \mathrm{d}Y_t=\big(-y_t+F_0(t)\big)\,\mathrm{d}t+\big(-z_t+G_0(t)\big)\,\mathrm{d}B_t+Z_t\,\mathrm{d}W_t, & Y_T=y_T+\varphi.\end{cases}\tag{4}$$

We have the following lemma:

Lemma 3.1. *For any* $x\in\mathbb{R}^n$, $(F_0,f_0,G_0,g_0)\in M^2(0,T;\mathbb{R}^{n+n+n\times l+n\times d})$, $\varphi\in L^2(\Omega,\mathcal{F}_T,P;\mathbb{R}^n)$, (4) *has a unique solution* (y,Y,z,Z) *in* $M^2(0,T;\mathbb{R}^{n+n+n\times l+n\times d})$.

Proof. The proof of uniqueness is similar to the one of Theorem 2.2 below. We only need to find a solution for (4). We consider the following linear backward doubly stochastic differential equations

$$\overline{Y}_t=\varphi-\int_t^T\big[\overline{Y}_s+F_0(s)-f_0(s)\big]\,\mathrm{d}s-\int_t^T\big[2\overline{Z}_s-g_0(s)\big]\,\mathrm{d}W_s-\int_t^T G_0(s)\,\mathrm{d}B_s.\tag{5}$$

By the result of [8], the above equation has a unique solution $(\overline{Y},\overline{Z})$. Then we can solve the following SDE

$$y_t=x+\int_0^t\big[-y_s-\overline{Y}_s+f_0(s)\big]\,\mathrm{d}s+\int_0^t\big[-\overline{Z}_s+g_0(s)\big]\,\mathrm{d}W_s-\int_0^t\bar z_s\,\mathrm{d}B_s.\tag{6}$$

Due to the result in [8], the above equation has a unique solution $(y,\bar z)$. And setting $Y=y+\overline{Y}$, $Z=\overline{Z}$, $z=\bar z$, we easily see that (y,Y,z,Z) is a solution to (4). Thus the existence is proved. □

The following a priori lemma is a key step in the proof of the method of continuation. It shows that, if for a fixed $\alpha=\alpha_0\in[0,1]$, (3) can be solved, then it can also be solved for $\alpha\in[\alpha_0,\alpha_0+\delta_0]$, for some positive constant δ_0 independent of α_0.

Lemma 3.2. *Under assumptions* (H1)–(H4), *there exists a positive constant* δ_0 *such that if, a priori, for some* $\alpha_0 \in [0,1)$, *and for each* $x \in \mathbb{R}^n$, $\varphi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n)$, $(F_0, f_0, G_0, g_0) \in M^2(0,T; \mathbb{R}^{n+n+n\times l+n\times d})$, (3) *has a unique solution, then for each* $\alpha \in [\alpha_0, \alpha_0 + \delta_0]$, *and* $x \in \mathbb{R}^n$, $\varphi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n)$, $(F_0, f_0, G_0, g_0) \in M^2(0,T; \mathbb{R}^{n+n+n\times l+n\times d})$, (3) *also has a unique solution in* $M^2(0,T; \mathbb{R}^{n+n+n\times l+n\times d})$.

Proof. Since for any $x \in \mathbb{R}^n$, $(F_0, f_0, G_0, g_0) \in M^2(0,T; \mathbb{R}^{n+n+n\times l+n\times d})$, $\varphi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n)$, there exists a unique solution to (3) for $\alpha = \alpha_0$, thus for each $\overline{U} = (\bar{y}, \overline{Y}, \bar{z}, \overline{Z}) \in M^2(0,T; \mathbb{R}^{n+n+n\times l+n\times d})$, there exists a unique quadruple $U = (y, Y, z, Z) \in M^2(0,T; \mathbb{R}^{n+n+n\times l+n\times d})$ satisfying the following equations

$$\begin{cases} \mathrm{d}y_t = \big[f^{\alpha_0}(t, U_t) + \delta\big(f(t, \overline{U}_t) + \overline{Y}_t\big) + f_0(t)\big]\,\mathrm{d}t - z_t\,\mathrm{d}B_t + \big[g^{\alpha_0}(t, U_t) + \delta\big(g(t, \overline{U}_t) + \overline{Z}_t\big) + g_0(t)\big]\,\mathrm{d}W_t, \\ \mathrm{d}Y_t = \big[F^{\alpha_0}(t, U_t) + \delta\big(F(t, \overline{U}_t) + \bar{y}_t\big) + F_0(t)\big]\,\mathrm{d}t + Z_t\,\mathrm{d}W_t + \big[G^{\alpha_0}(t, U_t) + \delta\big(G(t, \overline{U}_t) + \bar{z}_t\big) + G_0(t)\big]\,\mathrm{d}B_t, \\ y_0 = x, \quad Y_T = \Phi^{\alpha_0}(y_T) + \delta\big(\Phi(\bar{y}_T) - \bar{y}_T\big) + \varphi, \end{cases}$$

where δ is a positive number independent of α_0 and less than 1. We will prove that the mapping defined by $U = I_{\alpha_0+\delta}(\overline{U}): M^2(0,T; \mathbb{R}^{n+n+n\times l+n\times d}) \to M^2(0,T; \mathbb{R}^{n+n+n\times l+n\times d})$ is contractive for a small enough δ. Let $\overline{U}' = (\bar{y}', \overline{Y}', \bar{z}', \overline{Z}') \in M^2(0,T; \mathbb{R}^{n+n+n\times l+n\times d})$ and $U' = (y', Y', z', Z') = I_{\alpha_0+\delta}(\overline{U}')$, and

$$\begin{aligned} \widehat{\overline{U}} &= \overline{U} - \overline{U}' = \big(\hat{\bar{y}}, \widehat{\overline{Y}}, \hat{\bar{z}}, \widehat{\overline{Z}}\big) = \big(\bar{y} - \bar{y}', \overline{Y} - \overline{Y}', \bar{z} - \bar{z}', \overline{Z} - \overline{Z}'\big), \\ \widehat{U} &= U - U' = (\hat{y}, \widehat{Y}, \hat{z}, \widehat{Z}) = \big(y - y', Y - Y', z - z', Z - Z'\big). \end{aligned}$$

Applying Itô's formula to $\langle \hat{y}, \widehat{Y}\rangle$ on $[0,T]$ and by virtue of (H1) and (H4), we have, since $\mathbf{E}\hat{y}_0 = 0$

$$\begin{aligned} &\mathbf{E}\big\langle \hat{y}_T, \alpha_0\big(\Phi(y_T) - \Phi(y'_T)\big) + (1-\alpha_0)\hat{y}_T + \delta\big(\Phi(\bar{y}_T) - \Phi(\bar{y}'_T) - \hat{\bar{y}}_T\big)\big\rangle \\ &\quad \leqslant \mathbf{E}\int_0^T \bigg[(\alpha_0 - \mu\alpha_0 - 1)\big|\widehat{U}_t\big|^2 + \frac{\delta(k+1)}{2}\big|\widehat{U}_t\big|^2 + \frac{\delta(k+1)}{2}\big|\widehat{\overline{U}}_t\big|^2\bigg]\,\mathrm{d}t. \end{aligned}$$

Then by virtue of (H2) and (H4), we can derive that $[\theta - \frac{\delta(k+1)}{2}]\mathbf{E}\int_0^T |\widehat{U}_t|^2\,\mathrm{d}t \leqslant \frac{\delta(k+1)}{2}\mathbf{E}\int_0^T |\widehat{\overline{U}}_t|^2\,\mathrm{d}t + \delta(k+1)\,\mathbf{E}|\hat{y}_T||\hat{\bar{y}}_T|$, where $\theta = \min(1,\mu)$. Applying Itô's formula to $|\hat{y}|^2$ on $[0,T]$ and by virtue of (H4), by a standard method of estimation, we can derive that there exists a constant $c \geqslant 1$ which depends only on k, such that $\mathbf{E}|\hat{y}_T|^2 \leqslant c\mathbf{E}\int_0^T |\widehat{U}_t|^2\,\mathrm{d}t + \delta c\mathbf{E}\int_0^T |\widehat{\overline{U}}_t|^2\,\mathrm{d}t$. We now choose $\delta_0 = \frac{2\theta}{(4c+1)(c+1)(k+1)}$. Then for any $\delta \in [0,\delta_0]$, $\mathbf{E}\int_0^T |\widehat{U}_t|^2\,\mathrm{d}t \leqslant \frac{1}{4c}(\mathbf{E}\int_0^T |\widehat{\overline{U}}_t|^2\,\mathrm{d}t + \mathbf{E}|\hat{\bar{y}}_T|^2)$. Since $\delta_0 \leqslant \frac{1}{4c}$, we have for any $\delta \in [0,\delta_0]$, $\mathbf{E}|\hat{y}_T|^2 \leqslant \frac{1}{2}(\mathbf{E}\int_0^T |\widehat{\overline{U}}_t|^2\,\mathrm{d}t + \mathbf{E}|\hat{\bar{y}}_T|^2)$. It follows that, for each fixed $\delta \in [0,\delta_0]$, the mapping $I_{\alpha_0+\delta}$ is contractive in the following sense $\mathbf{E}\int_0^T |\widehat{U}_t|^2\,\mathrm{d}t + \mathbf{E}|\hat{y}_T|^2 \leqslant \frac{3}{4}(\mathbf{E}\int_0^T |\widehat{\overline{U}}_t|^2\,\mathrm{d}t + \mathbf{E}|\hat{\bar{y}}_T|^2)$. Thus this mapping has a unique fixed point $U = (y, Y, z, Z)$ in $M^2(0,T; \mathbb{R}^{n+n+n\times l+n\times d})$, which is the solution of (3) for $\alpha = \alpha_0 + \delta$, as $\delta \in [0,\delta_0]$. The proof is complete. □

4. The proof of Theorem 2.2

Now we can give the proof of Theorem 2.2 – the existence and uniqueness theorem of (2).

Proof. *Uniqueness*. Let $U = (y, Y, z, Z)$ and $U' = (y', Y', z', Z')$ be two solutions of (2). We use the same notations as in Lemma 3.2. Applying Itô's formula to $\langle \hat{y}, \widehat{Y}\rangle$ on $[0,T]$, we have $\mathbf{E}\langle \hat{y}_T, \widehat{\Phi}(y_T)\rangle = \mathbf{E}\int_0^T \langle A(t, U_t) - A(t, U'_t), \widehat{U}_t\rangle\,\mathrm{d}t$. By virtue of (H1) and (H2), it follows that $\mu\mathbf{E}\int_0^T |U_t - U'_t|^2\,\mathrm{d}t \leqslant 0$. Thus $U = U'$. The uniqueness is proved.

Existence. By Lemma 3.1, for any $x \in \mathbb{R}^n$, $(F_0, f_0, G_0, g_0) \in M^2(0,T; \mathbb{R}^{n+n+n\times l+n\times d})$, $\varphi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R}^n)$, (3) has a unique solution in $M^2(0,T; \mathbb{R}^{n+n+n\times l+n\times d})$ as $\alpha = 0$.

It follows from Lemma 3.2 that there exists a positive constant $\delta_0 = \delta_0(k,\mu)$ such that for any $\delta \in [0,\delta_0]$ and $x \in \mathbb{R}^n$, $\varphi \in L^2(\Omega,\mathcal{F}_T,P;\mathbb{R}^n)$, $(F_0, f_0, G_0, g_0) \in M^2(0,T;\mathbb{R}^{n+n+n\times l+n\times d})$, (3) has a unique solution for $\alpha=\delta$. Since δ_0 depends only on k and μ, we can repeat this process for N times with $1 \leqslant N\delta_0 < 1+\delta_0$. In particular, for $\alpha = 1$ with $(F_0, f_0, G_0, g_0) \equiv 0$ and $\varphi \equiv 0$, (3) has a unique solution in $M^2(0,T;\mathbb{R}^{n+n+n\times l+n\times d})$. The proof is complete. □

Remark 1. In the case where the initial condition $y_0 = x \in L^2(\Omega,\mathcal{F}_0,P;\mathbb{R}^n)$, all the results in this paper still hold true.

5. Example: a doubly stochastic Hamiltonian system

Consider the following doubly stochastic Hamiltonian system

$$\begin{cases} \mathrm{d}y_t = H_Y\,\mathrm{d}t + H_Z\,\mathrm{d}W_t - z_t\,\mathrm{d}B_t, & y_0 = x, \\ \mathrm{d}Y_t = -H_y\,\mathrm{d}t - H_z\,\mathrm{d}B_t + Z_t\,\mathrm{d}W_t, & Y_T = \Phi_y(y_T), \end{cases} \tag{7}$$

where $H(y,Y,z,Z):\mathbb{R}^4 \to R$, $\Phi(y): R \to R$; $H_Y \doteq \nabla_Y H$, $\Phi_y \doteq \nabla_y \Phi$. The Brownian motions $\{W_t;\ 0 \leqslant t \leqslant T\}$ and $\{B_t;\ 0 \leqslant t \leqslant T\}$ are both assumed to be 1-dimensional. Assume that both the derivatives of 2-order of H and the derivatives of 1-order of Φ are bounded, H is concave on (Y,Z) and convex on (y,z) in the following sense ($\mu > 0$):

$$\begin{bmatrix} -H_{yy} & -H_{yY} & -H_{yz} & -H_{yZ} \\ H_{Yy} & H_{YY} & H_{Yz} & H_{YZ} \\ -H_{zy} & -H_{zY} & -H_{zz} & -H_{zZ} \\ H_{Zy} & H_{ZY} & H_{Zz} & H_{ZZ} \end{bmatrix} \leqslant -\mu\mathbf{I}, \quad \forall (y,Y,z,Z) \in \mathbb{R}^4,$$

and Φ is convex on (y): $\Phi_{yy} \geqslant 0$, $\forall y \in \mathbb{R}$. By Theorem 2.2, we know that this doubly stochastic Hamiltonian system (7) has a unique solution (y,Y,z,Z) in $M^2(0,T;\mathbb{R}^4)$.

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