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Number Theory

Complex Pisot numbers of small modulus

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Abstract

We use an algorithm of Chamfy to determine all complex Pisot numbers of modulus less than 1.17. ***To cite this article: D. Garth, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Nombres de Pisot imaginaires de petit module. Nous utilisons une méthod de Chamfy pour déterminer les nombres de Pisot imaginaires de module au plus 1.17. ***Pour citer cet article : D. Garth, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

1. Introduction

A PV number, or Pisot–Vijayaraghavan number, is an algebraic integer $\alpha > 1$ whose remaining conjugates lie in the open unit disk. It is well known that the smallest PV number is $\theta_0 \approx 1.3247$, the positive zero of $z^3 - z - 1$ [9]. In [5] Dufresnoy and Pisot proved that the smallest limit point of the set S_1 of Pisot numbers is $\frac{1+\sqrt{5}}{2}$. In [6] they developed a powerful algorithm which they used to classify all the Pisot numbers less than $\frac{1+\sqrt{5}}{2}$.

A complex PV number is a non-real algebraic integer α, with $|\alpha| > 1$, whose remaining conjugates other than $\bar{\alpha}$ lie in the open unit disk. Without loss of generality, we may require the real part of a complex PV number to be non-negative. Chamfy has shown that the smallest complex PV number has modulus $\sqrt{\theta_0} \approx 1.1509$, with either $z^3 - z^2 + 1$ or $z^6 - z^2 + 1$ as a minimal polynomial [4]. In light of this, it is natural to ask whether the smallest limit point of S_2, the set of complex Pisot numbers, is the one with minimal polynomial $z^4 + z^2 - 1$, having modulus $\sqrt{(1+\sqrt{5})/2} \approx 1.2720$.

Toward answering this question, Chamfy has generalized Dufresnoy and Pisot's algorithm [4], although she didn't use this algorithm in her study of complex PV numbers. In this paper we discuss how to use Chamfy's algorithm to classify small complex PV numbers. Our main result is the following.

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Theorem 1.1. *There are no limit points of the set of complex Pisot numbers of modulus less than* 1.17.

Using the algorithm we were also able to determine all the complex PV numbers of modulus less than 1.17. The algorithm is also useful for finding limit points of S_2. This is made possible, in part, by the following theorem of Cantor (see Theorem 5.7 of [3]). The theorem is a generalization of a characterization of the limit points of the Pisot numbers discovered by Dufresnoy and Pisot in [5].

Theorem 1.2. *A complex PV number θ with minimal polynomial $P(z)$ is a limit point of S_2 if and only if there exists a nonzero polynomial $A(z)$, with integer coefficients, such that $|A(z)| \leqslant |P(z)|$ whenever $|z| = 1$, with equality holding in at most finitely many points.*

The main idea in the proof of this theorem is to apply Rouche's theorem to the function $P_n(z) = A(z) + \lambda z^n P(z)$, letting $\lambda \to 1$. As $n \to \infty$, the roots of $P_n(z)$ approach those of $P(z)$. The following discussion of Chamfy's algorithm will reveal that many complex PV numbers can be expressed using this construction.

2. The fundamental correspondence

In the study of the Pisot numbers, much use has been made of an association between S_1 and a certain set of rational functions. See [2] for a good explanation of this. We will describe a similar association for the set of complex PV numbers. Let θ be a complex PV number with minimal polynomial $P(z)$ of degree d, and assume that $P(0) > 0$. Define $Q(z) = \varepsilon z^d P(\frac{1}{z})$, where $\varepsilon = \pm 1$ is such that $Q(0) = 1$. Suppose $A(z) \neq \pm Q(z)$ is a polynomial with integer coefficients which satisfies $A(0) > 0$ and $|A(z)| \leqslant |Q(z)|$ for $|z| = 1$. We then say that the rational function $f = A/Q$ is *associated* with θ.

If $P(z) \neq \pm Q(z)$ we can let $A = P$. In this case we say that $f = A/Q$ has *rank d*. In the event that $P = \pm Q$, it follows that θ is a limit point of S_2 [8]. The existence of the appropriate A in this case is guaranteed by Theorem 1.2. We say that such f have *infinite rank*. Using the notation of [1], we let N_2 denote the set of all f which are associated to a $\theta \in S_2$. Notice that each $f(z) \in N_2$ is meromorphic with 2 simple poles in $|z| < 1$, and $|f(z)| \leqslant 1$ whenever $|z| = 1$. Also, suppose

$$f(z) = u_0 + u_m z^m + u_{m+m'} z^{m+m'} + \cdots + u_n z^n + \cdots \tag{1}$$

is the Taylor series expansion of f in a neighborhood of the origin, where $u_m \neq 0$ and $u_{m+m'} \neq 0$. Since $Q(0) = 1$, it follows that the coefficients u_n in the expansion are integers.

3. Inequalities for the coefficients of $f \in N_2$

For the remainder of this paper, let $\theta \in S_2$ and assume that $|\theta| < 1.17$. Let $f = \frac{A}{Q} \in N_2$ be associated with θ, and let α and β be the poles of f in $|z| < 1$. Then α and β lie in the right half plane and satisfy $\frac{1}{1.17} < |\alpha|, |\beta| < 1$. Define

$$\phi(z) = \frac{A(z)}{Q(z)} \cdot \frac{\alpha - z}{1 - \alpha z} \cdot \frac{\beta - z}{1 - \beta z}.$$

Then $\phi(z)$ is analytic and bounded by 1 on the closed unit disk. Let $\phi(z) = \sum d_i z^i$ be the Taylor expansion of ϕ in the closed unit disk. Then Lemma 3.5.1 of [1] says that for $i \geqslant 1$ the following inequalities hold:

$$|d_i| \leqslant 1 - d_0^2 \quad \text{and} \quad -\left(1 - d_0^2 - \frac{d_i^2}{1 + d_i}\right) \leqslant d_{2i} \leqslant 1 - d_0^2 - \frac{d_i^2}{1 - d_i}. \tag{2}$$

These inequalities give rise to inequalities among α, β, and the coefficients u_n. Unfortunately, however, they quickly become unwieldy. Chamfy's algorithm gives us a different set of inequalities.

Suppose f is of rank s, possibly infinite. Chamfy proved the existence of an integer n_0 such that for $n_0 \leqslant n \leqslant s$, the coefficients u_n of the Taylor expansion of f satisfy a system of inequalities of the form

$$v_n^-(u_0, \ldots, u_{n-1}) \leqslant u_n \leqslant v_n^+(u_0, \ldots, u_{n-1}). \tag{3}$$

We determine v_n^- by finding a pair of relatively prime polynomials $d_n^-(z)$ and $e_n^-(z) \in \mathbb{Q}[z]$ of degree n, with $d_n^-(z) = -z^n e_n^-(\frac{1}{z})$, with the property that, in a neighborhood of the origin,

$$\frac{d_n^-(z)}{e_n^-(z)} = u_0 + u_1 z + \cdots + u_{n-1} z^{n-1} + v_n^- z^n + \cdots. \tag{4}$$

Similarly, we obtain v_n^+ by finding a pair of polynomials $d_n^+(z)$ and $e_n^+(z) \in \mathbb{Q}[z]$ with $d_n^+(z) = z^n e_n^+(\frac{1}{z})$ such that, in a neighborhood of the origin,

$$\frac{d_n^+(z)}{e_n^+(z)} = u_0 + u_1 z + \cdots + u_{n-1} z^{n-1} + v_n^+ z^n + \cdots. \tag{5}$$

Moreover, if $v_n^- = u_n$ for some n, then $f = d_n^-(z)/e_n^-(z)$. Similarly, if $v_n^+ = u_n$ then $f = d_n^+(z)/e_n^+(z)$.

To effectively compute n_0, Chamfy's algorithm subjects f to one of three transformations. For our purposes, it suffices to use the single transformation

$$\gamma(z) = \delta z^k \frac{(z^{2m} + u_m z^m - 1) f(z) - (z^{2m} - 1)}{(z^{2m} - 1) f(z) - (z^{2m} - u_m z^m - 1)},$$

where $k \in \mathbb{Z}$ is chosen so that $\gamma(z)$ has no zeros or poles at the origin, and $\delta = \pm 1$ is chosen so that $\gamma(0) > 0$. Let $\{c_\ell\}$ be the coefficient sequence of the Taylor expansion of γ in a neighborhood of the origin. γ has at most one pole in $|z| < 1$, and $|\gamma(z)| \leqslant 1$ whenever $|z| = 1$. When γ is analytic in $|z| \leqslant 1$, Chamfy established, for $\ell \geqslant 1$, the existence of a set of inequalities for $\{c_\ell\}$ of a form similar to (3) [4]. These are obtained from two unique pairs of polynomials D_ℓ^-, E_ℓ^- and D_ℓ^+, E_ℓ^+ possessing properties similar to (4) and (5) above in relation to γ. From these polynomials we obtain $d_n^\pm(z)$, $e_n^\pm(z)$ [4]. n_0 is then the degree of the $d_n^\pm(z)$ polynomial corresponding to the $D_\ell^\pm(z)$ of smallest degree. If γ has one pole in $|z| < 1$, Dufresnoy and Pisot proved the inequalities exist for $\ell \geqslant 2$ if $c_0 \neq 1$ and for $\ell \geqslant 3$ if $c_0 = 1$ (see Theorem 7.1.4 of [1]). They also showed that $c_0 \geqslant 1$ and $c_1 \geqslant \max(1, c_0^2 - 1)$. This fact enables us to improve on Chamfy's inequalities in (3) by getting one sided bounds for u_n in some cases when $n < n_0$.

4. Results

Given that $\frac{1}{1.17} < |\alpha|, |\beta| < 1$, we used the inequalities in (2) to bound the first 6 coefficients u_n of f. We then computed the values of n_0 for the functions f whose first few Taylor coefficients were within these bounds. The inequalities in (3) give rise to a coefficient tree with the possible integer values of u_n as the nodes. The terminal nodes in the tree for which the $d_n^\pm(z)$ have integer coefficients yield complex PV numbers. Paths to infinity in the tree correspond to $f \in N_2$ of infinite rank, and hence to limit points of S_2. For these paths the $d_n^\pm(z)$ polynomials are of the form $\tilde{A}(z) \pm z^n \widetilde{P}(z)$, where $\widetilde{P}$ is the minimal polynomial of a limit point of S_2, and $\tilde{A}$ is the corresponding polynomial guaranteed by Theorem 1.2. Table 1 lists the 10 smallest limit points of S_2 that we found, along with the corresponding $\widetilde{P}$ and $\tilde{A}$.

In [7] we generated a list of complex PV numbers less than $\sqrt{(1+\sqrt{5})/2}$ in modulus. The list contains 72 numbers with non-zero imaginary parts. Complex PV numbers on the imaginary axis are just the square roots of the PV numbers in modulus. Thus, all purely imaginary complex PV numbers less than $\sqrt{(1+\sqrt{5})/2}$ in modulus are known. The 10 smallest complex PV numbers on our list are in Table 2. The table also shows the minimal polynomial $P(z)$ along with the construction $P(z) = \tilde{A}(z) \pm z^n \widetilde{P}(z)$ alluded to above. $P(z) = \tilde{A}(z) \pm z^n \widetilde{P}(z)$ alluded to above.

Table 1
Small limit points of complex Pisot numbers

Tableau 1
Nombres de Pisot imaginaires de petit module de l'ensemble dérivé

$\|\theta\|$	$\widetilde{P}(z)$	$\tilde{A}(z)$
1.2720	$1 - z^2 - z^4$	$1 - z^4$
1.3122	$1 + z - z^2 - z^3 + z^4$	$1 - z^2 + z^4$
1.3247	$1 + z^2 + 2z^4 + z^6$	$1 + z^2 + z^4 + z^6$
1.3353	$1 - z - z^2 + z^3 - z^4$	$1 - z + z^3 - z^4$
1.3453	$1 - z^2 - z^3 - z^5$	$1 - z^5$
1.3497	$1 + z - z^3 - z^4 + z^5$	$1 - z^2 - z^3 + z^5$
1.3562	$1 + z - z^2 + z^3$	$1 + z^3$
1.3641	$1 + 2z + z^2 + z^5$	$1 + z + z^4 + z^5$
1.3722	$1 + z - z^3 + z^4$	$1 + z^4$
1.4098	$1 - 2z^2 + z^4 - z^5$	$1 - z^2 + z^3 + z^4 - z^5$

Table 2
Complex Pisot numbers of small modulus

Tableau 2
Nombres de Pisot imaginaires de petit module

$\|\theta\|$	$P(z)$	$\tilde{A}(z) \pm z^n \widetilde{P}(z)$
1.1509	$1 - z^2 + z^3$	$[(1 + z^3) - z(1 + z - z^2 + z^3)]/(1 - z)$
1.1509	$1 - z^2 + z^6$	$(1 - z^4) - z^2(1 - z^2 - z^4)$
1.1617	$1 - z + z^3 - z^4 + z^5$	$(1 - z^4) - z(1 - z^2 - z^4)$
1.1661	$1 + z - z^3 - z^4 + z^6$	$(1 - z^2 - z^3 + z^5) + z(1 + z - z^3 - z^4 + z^5)$
1.1679	$1 - z^2 - z^5$	$(1 - z + z^3 - z^4) + z(1 - z - z^2 + z^3 - z^4)$
1.1744	$1 - z^6 + z^7$	$(1 - z^2 - z^3 + z^5) + z^2(1 + z - z^3 - z^4 + z^5)$
1.1748	$1 - z^6 - z^8$	$(1 - z^4) + z^4(1 - z^2 - z^4)$
1.1822	$1 - z^5 + z^6$	$[(1 - z^2 + z^4) - z^4(1 + z - z^2 - z^3 + z^4)]/(1 - z^2)$
1.1837	$1 - z^3 + z^4$	$[(1 + z^4) - z^2(1 + z - z^3 + z^4)]/(1 - z^2)$
1.1837	$1 + z + z^4$	$[(1 + z^3) - z^2(1 + z - z^2 + z^3)]/(1 - z)$

For PV numbers, there is an ordering relation among the zeros of the $d_n^{\pm}(z)$ polynomials which provides a nice stopping condition on Dufresnoy and Pisot's algorithm [2]. Chamfy's algorithm has no such condition, so we used a combination of the inequalities (2) and (3) which enabled us to consider only finite trees. This enabled us to show that the first half of the list in Table 2 is complete. Numerous examples of $f \in N_2$ exist for which n_0 can be arbitrarily large. This was the major obstacle in our attempts to prove that any more of the list is complete.

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