## Differential Geometry

# Spin ${ }^{c}$-manifolds and elliptic genera 

Fei Han, Weiping Zhang<br>Nankai Institute of Mathematics, Nankai University, Tianjin 300071, PR China

Received 16 February 2003: accepted after revision 9 May 2003
Presented by Jean-Michel Bismut


#### Abstract

We present an extension of the "miraculous cancellation" formulas of Alvarez-Gaumé, Witten and Kefeng Liu to a twisted version where an extra complex line bundle is involved. Relations to the Ochanine congruence formula on $8 k+4$ dimensional Spin $^{c}$ manifolds are discussed. To cite this article: F. Han, W. Zhang, C. R. Acad. Sci. Paris, Ser. I 336 (2003). © 2003 Académie des sciences. Published by Éditions scientifiques et médicales Elsevier SAS. All rights reserved.


## Résumé

Variétés Spin ${ }^{c}$ et genre elliptique. Nous présentons une extension de formules d'annulation d'Alvarez-Gaumé, Witten et Liu lorsqu'on tensorise les fibrés considérés par un fibré en droites complexe. On discute le lien entre nos formules et les formules de congruence d'Ochanine pour les variétés $\operatorname{Spin}^{c}$ de dimension $8 k+4$. Pour citer cet article:F. Han, W. Zhang, C. R. Acad. Sci. Paris, Ser. I 336 (2003).
© 2003 Académie des sciences. Published by Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

## 1. Introduction

Let $M$ be a Riemannian manifold. Let $\nabla^{T M}$ be the associated Levi-Civita connection, and let $R^{T M}=\nabla^{T M, 2}$ be the curvature of $\nabla^{T M}$. Then $\nabla^{T M}$ extends canonically to a Hermitian connection $\nabla^{T_{\mathbf{C}} M}$ on $T_{\mathbf{C}} M=T M \otimes \mathbf{C}$.

Let $\hat{A}\left(T M, \nabla^{T M}\right), \hat{L}\left(T M, \nabla^{T M}\right)$ be the Hirzebruch characteristic forms defined by

$$
\begin{equation*}
\hat{A}\left(T M, \nabla^{T M}\right)=\operatorname{det}^{1 / 2}\left(\frac{(\sqrt{-1} / 4 \pi) R^{T M}}{\sinh \left((\sqrt{-1} / 4 \pi) R^{T M}\right)}\right), \quad \hat{L}\left(T M, \nabla^{T M}\right)=\operatorname{det}^{1 / 2}\left(\frac{(\sqrt{-1} / 2 \pi) R^{T M}}{\tanh \left((\sqrt{-1} / 4 \pi) R^{T M}\right)}\right) \tag{1}
\end{equation*}
$$

and let $\operatorname{ch}\left(T_{\mathbf{C}} M, \nabla^{T_{\mathbf{C}} M}\right)$ denote the Chern character form associated to $\left(T_{\mathbf{C}} M, \nabla^{T_{\mathbf{C}} M}\right.$ ) (cf. [9, Section 1.6]).
When $\operatorname{dim} M=12$, the following equation for 12 -forms was proved by Alvarez-Gaumé and Witten in [1], which they called "miraculous cancellation",

$$
\begin{equation*}
\left\{\hat{L}\left(T M, \nabla^{T M}\right)\right\}^{(12)}=\left\{8 \hat{A}\left(T M, \nabla^{T M}\right) \operatorname{ch}\left(T_{\mathbf{C}} M, \nabla^{T_{\mathbf{C}} M}\right)-32 \hat{A}\left(T M, \nabla^{T M}\right)\right\}^{(12)} \tag{2}
\end{equation*}
$$

These authors also discussed the applications of such formulas to physics.

[^0]In [5], Kefeng Liu generalized (2) to arbitrarily $8 k+4$ dimensional manifolds by developing modular invariance properties of characteristic numbers.

In this Note, we present an extension of Liu's formula in the presence of an extra complex line bundle (or equivalently, a rank two real oriented vector bundle). In dimension 12, this extension can be described as follows: let $\xi$ be a rank two real oriented Euclidean vector bundle, equipped with a Euclidean connection $\nabla^{\xi}$, let $c=e\left(\xi, \nabla^{\xi}\right)$ be the associated Euler form (cf. [9, Section 3.4]). Then the following equation for 12 -forms holds,

$$
\begin{align*}
\left\{\frac{\hat{L}\left(T M, \nabla^{T M}\right)}{\cosh ^{2}(c / 2)}\right\}^{(12)}= & \left\{\left[8 \hat { A } ( T M , \nabla ^ { T M } ) \operatorname { c h } \left(T_{\mathbf{C}} M, \nabla^{T} \mathbf{C} M\right.\right.\right. \\
& -32 \hat{A}\left(T M, \nabla^{T M}\right)  \tag{3}\\
& \left.\left.-24 \hat{A}\left(T M, \nabla^{T M}\right)\left(\mathrm{e}^{c}+\mathrm{e}^{-c}-2\right)\right] \cosh \left(\frac{c}{2}\right)\right\}^{(12)}
\end{align*}
$$

Clearly, when $\xi$ is trivial and $c=0$, (3) reduces to the formula (2) of Alvarez-Gaumé and Witten. Our work was motivated by the Ochanine congruence formula [7].

## 2. Main results

Let $M$ be a $8 k+4$ dimensional Riemannian manifold. Let $\nabla^{T M}$ be the associated Levi-Civita connection. Let $V$ be a rank $2 l$ real Euclidean vector bundle over $M$ equipped with a Euclidean connection $\nabla^{V}$. Let $\xi$ be a rank two real oriented Euclidean vector bundle over $M$ carrying with a Euclidean connection $\nabla^{\xi}$. Let $c=e\left(\xi, \nabla^{\xi}\right)$ be the Euler form of $\xi$ canonically associated to $\nabla^{\xi}$.

Set $V_{\mathbf{C}}=V \otimes \mathbf{C}$ and $\xi_{\mathbf{C}}=\xi \otimes \mathbf{C}$. Then $V_{\mathbf{C}}$ and $\xi_{\mathbf{C}}$ are complex vector bundles over $M$, each of which is equipped with a Hermitian metric, and a unitary connection.

If $W$ is a Hermitian vector bundle over $M$ equipped with a Hermitian connection $\nabla^{W}$, we denote by $\operatorname{ch}\left(W, \nabla^{W}\right)$ the associated Chern character form (cf. [9, Section 1.6]). Also, for any complex number $t$, set $\Lambda_{t}(W)=$ $\left.\mathbf{C}\right|_{M}+t W+t^{2} \Lambda^{2}(W)+\cdots$ and $S_{t}(W)=\left.\mathbf{C}\right|_{M}+t W+t^{2} S^{2}(W)+\cdots$, where for any integer $j \geqslant 1, \Lambda^{j}(W)$ (resp. $S^{j}(W)$ ) is the $j$-th exterior (resp. symmetric) power of $W$. Set $\widetilde{W}=W-\mathbf{C}^{\mathrm{rk}(W)}$.

Let $q=\mathrm{e}^{2 \pi \sqrt{-1} \tau}$ with $\tau \in \mathbf{H}$, the upper half plane. Set

$$
\begin{align*}
& \Theta_{1}\left(M, V_{\mathbf{C}}, \xi_{\mathbf{C}}\right)=\bigotimes_{n=1}^{\infty} S_{q^{n}}\left(\widetilde{T_{\mathbf{C}} M}\right) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{q^{m}}\left(\tilde{V}_{\mathbf{C}}-2 \tilde{\xi}_{\mathbf{C}}\right) \otimes \bigotimes_{r=1}^{\infty} \Lambda_{q^{r-1 / 2}}\left(\tilde{\xi}_{\mathbf{C}}\right) \otimes \bigotimes_{s=1}^{\infty} \Lambda_{-q^{s-1 / 2}}\left(\tilde{\xi}_{\mathbf{C}}\right)  \tag{4}\\
& \Theta_{2}\left(M, V_{\mathbf{C}}, \xi_{\mathbf{C}}\right)=\bigotimes_{n=1}^{\infty} S_{q^{n}}\left(\widetilde{T_{\mathbf{C}} M}\right) \otimes \bigotimes_{m=1}^{\infty} \Lambda_{-q^{m-1 / 2}}\left(\widetilde{V}_{\mathbf{C}}-2 \tilde{\xi}_{\mathbf{C}}\right) \otimes \bigotimes_{r=1}^{\infty} \Lambda_{q^{r-1 / 2}}\left(\tilde{\xi}_{\mathbf{C}}\right) \otimes \bigotimes_{s=1}^{\infty} \Lambda_{q^{s}}\left(\tilde{\xi}_{\mathbf{C}}\right) \tag{5}
\end{align*}
$$

Clearly, $\Theta_{1}\left(M, V_{\mathbf{C}}, \xi_{\mathbf{C}}\right)$ and $\Theta_{2}\left(M, V_{\mathbf{C}}, \xi_{\mathbf{C}}\right)$ admit formal Fourier expansion in $q^{1 / 2}$ as

$$
\begin{align*}
& \Theta_{1}\left(M, V_{\mathbf{C}}, \xi_{\mathbf{C}}\right)=A_{0}\left(M, V_{\mathbf{C}}, \xi_{\mathbf{C}}\right)+A_{1}\left(M, V_{\mathbf{C}}, \xi_{\mathbf{C}}\right) q^{1 / 2}+\cdots  \tag{6}\\
& \Theta_{2}\left(M, V_{\mathbf{C}}, \xi_{\mathbf{C}}\right)=B_{0}\left(M, V_{\mathbf{C}}, \xi_{\mathbf{C}}\right)+B_{1}\left(M, V_{\mathbf{C}}, \xi_{\mathbf{C}}\right) q^{1 / 2}+\cdots \tag{7}
\end{align*}
$$

where the $A_{j}$ 's and $B_{j}$ 's are elements in the semi-group generated by Hermitian vector bundles over $M$. These vector bundles $A_{j}, B_{j}$ are naturally equipped with Hermitian metrics and unitary connections $\nabla^{A_{j}}, \nabla^{B_{j}}$.

Let $R^{V}=\nabla^{V, 2}$ denote the curvature of $\nabla^{V}$. We can now state our main result as follows.
Theorem 2.1. If the equality for the first Pontryagin forms $p_{1}\left(T M, \nabla^{T M}\right)=p_{1}\left(V, \nabla^{V}\right)$ holds, then one has the equation for $(8 k+4)$-forms,

$$
\begin{equation*}
\left\{\frac{\hat{A}\left(T M, \nabla^{T M}\right) \operatorname{det}^{1 / 2}\left(2 \cosh \left((\sqrt{-1} / 4 \pi) R^{V}\right)\right)}{\cosh ^{2}(c / 2)}\right\}^{(8 k+4)}=2^{l+2 k+1} \sum_{r=0}^{k} 2^{-6 r}\left\{b_{r} \cosh \left(\frac{c}{2}\right)\right\}^{(8 k+4)} \tag{8}
\end{equation*}
$$

where each $b_{r}, 0 \leqslant r \leqslant k$, is a finite canonical integral linear combination of the characteristic forms $\hat{A}\left(T M, \nabla^{T M}\right) \operatorname{ch}\left(B_{j}, \nabla^{B_{j}}\right), j \geqslant 0$.

When $\xi=\mathbf{R}^{2}$ and $c=0$, Theorem 2.1 is exactly Liu's result in [5, Theorem 1].
If we take $V=T M$ and $\nabla^{V}=\nabla^{T M}$ in (8), we get

$$
\begin{equation*}
\left\{\frac{\hat{L}\left(T M, \nabla^{T M}\right)}{\cosh ^{2}(c / 2)}\right\}^{(8 k+4)}=8 \sum_{r=0}^{k} 2^{6 k-6 r}\left\{b_{r} \cosh \left(\frac{c}{2}\right)\right\}^{(8 k+4)} \tag{9}
\end{equation*}
$$

In the case where $k=1$, one obtains (3) from (9).
Now assume that $M$ is closed, oriented and carries a $\operatorname{Spin}^{c}$ structure with $[c] \equiv w_{2}(T M) \bmod 2$, where $w_{2}(T M)$ is the second Stiefel-Whitney class of $T M$. Let $B$ be a connected closed oriented $8 k+2$ submanifold in $M$ such that $[B] \in H_{8 k+2}(M, \mathbf{Z})$ is Poincaré dual to [c]. Let $B \cdot B$ be the self-intersection of $B$ in $M$ which can be thought of as a closed oriented $8 k$ manifold. Then by [7], we know that

$$
\begin{equation*}
\int_{M} \frac{\hat{L}\left(T M, \nabla^{T M}\right)}{\cosh ^{2}(c / 2)}=\operatorname{Sign}(M)-\operatorname{Sign}(B \cdot B) \tag{10}
\end{equation*}
$$

On the other hand, by [2], we know that each $\int_{M} b_{r} \cosh \left(\frac{c}{2}\right), 0 \leqslant r \leqslant k$, is an integer. Combining this argument with (9) and (10), we deduce that

$$
\begin{equation*}
\frac{\operatorname{Sign}(M)-\operatorname{Sign}(B \cdot B)}{8} \equiv \int_{M} b_{k} \cosh \left(\frac{c}{2}\right) \quad \bmod 64 \tag{11}
\end{equation*}
$$

By combining (11) with the Rokhlin type congruences proved in [8], one can give a direct proof of the analytic version of the Ochanine congruence formula [7] stated in [6]. Moreover, if $M$ is spin, then using (11), the Ochanine divisibility result (cf. [7]) and [2], we get

$$
\begin{equation*}
\operatorname{Sign}(B \cdot B) \equiv 0 \bmod 8 \tag{12}
\end{equation*}
$$

a result which seems to be of interest by itself.
On the other hand, there are twisted cancellation formulas similar to (8), (9) on $8 k$ manifolds, generalizing the (untwisted) cancellation formulas stated in [5, p. 32].

More details and further applications will be given in [4].

## 3. Proof of Theorem 2.1

The methods of [5, Section 3] can be adapted here, with obvious modifications which take into account the presence of $\xi$ and $c$. Here, we only indicate the main steps of the proof.

First, since (8) is a local assertion, we may and we will assume that both $T M$ and $V$ are oriented. As in [5], we use the notation of formal Chern roots $\left\{ \pm 2 \pi \sqrt{-1} y_{v}\right\}$ and $\left\{ \pm 2 \pi \sqrt{-1} x_{j}\right\}$ for $\left(V_{\mathbf{C}}, \nabla^{\mathbf{C}}\right)$ and $\left(T_{\mathbf{C}} M, \nabla^{T_{\mathbf{C}}{ }^{M}}\right)$ respectively. We also set $c=2 \pi \sqrt{-1} u$.

For $\tau \in \mathbf{H}$ and $q=\mathrm{e}^{2 \pi \sqrt{-1} \tau}$, set

$$
\begin{align*}
& P_{1}(\tau)=\left\{\frac{\hat{A}\left(T M, \nabla^{T M}\right) \operatorname{det}^{1 / 2}\left(2 \cosh \left((\sqrt{-1} / 4 \pi) R^{V}\right)\right)}{\cosh ^{2}(c / 2)} \operatorname{ch}\left(\Theta_{1}\left(M, V_{\mathbf{C}}, \xi_{\mathbf{C}}\right), \nabla^{\Theta_{1}\left(M, V_{\mathbf{C}}, \xi_{\mathbf{C}}\right)}\right)\right\}^{(8 k+4)},  \tag{13}\\
& P_{2}(\tau)=\left\{\hat{A}\left(T M, \nabla^{T M}\right) \operatorname{ch}\left(\Theta_{2}\left(M, V_{\mathbf{C}}, \xi_{\mathbf{C}}\right), \nabla^{\Theta_{2}\left(M, V_{\mathbf{C}}, \xi_{\mathbf{C}}\right)}\right) \cosh \left(\frac{c}{2}\right)\right\}^{(8 k+4)} \tag{14}
\end{align*}
$$

where $\nabla^{\Theta_{i}\left(M, V_{\mathbf{C}}, \xi_{\mathbf{C}}\right)}, i=1,2$, are the induced Hermitian connections with $q^{1 / 2}$-coefficients on $\Theta_{i}\left(M, V_{\mathbf{C}}, \xi_{\mathbf{C}}\right)$ one gets from the $\nabla^{A_{j}}, \nabla^{B_{j}}$ 's (compare with (6) and (7)). Then a direct computation shows that

$$
\begin{align*}
& P_{1}(\tau)=2^{l}\left\{\prod_{j=1}^{4 k+2}\left(x_{j} \frac{\theta^{\prime}(0, \tau)}{\theta\left(x_{j}, \tau\right)}\right)\left(\prod_{v=1}^{l} \frac{\theta_{1}\left(y_{v}, \tau\right)}{\theta_{1}(0, \tau)}\right) \frac{\theta_{1}^{2}(0, \tau)}{\theta_{1}^{2}(u, \tau)} \frac{\theta_{3}(u, \tau)}{\theta_{3}(0, \tau)} \frac{\theta_{2}(u, \tau)}{\theta_{2}(0, \tau)}\right\}^{(8 k+4)},  \tag{15}\\
& P_{2}(\tau)=\left\{\prod_{j=1}^{4 k+2}\left(x_{j} \frac{\theta^{\prime}(0, \tau)}{\theta\left(x_{j}, \tau\right)}\right)\left(\prod_{v=1}^{l} \frac{\theta_{2}\left(y_{v}, \tau\right)}{\theta_{2}(0, \tau)}\right) \frac{\theta_{2}^{2}(0, \tau)}{\theta_{2}^{2}(u, \tau)} \frac{\theta_{3}(u, \tau)}{\theta_{3}(0, \tau)} \frac{\theta_{1}(u, \tau)}{\theta_{1}(0, \tau)}\right\}^{(8 k+4)}, \tag{16}
\end{align*}
$$

where $\theta(z, \tau)$ and $\theta_{i}(z, \tau), i=1,2,3$, are the classical Jacobi theta functions (cf. [3]).
Since $p_{1}\left(T M, \nabla^{T M}\right)=p_{1}\left(V, \nabla^{V}\right)$, i.e., $\sum_{j=1}^{4 k+2} x_{j}^{2}=\sum_{v=1}^{l} y_{v}^{2}$, by (15), (16) and by the transformation laws of theta functions (cf. [3]), one verifies directly that $P_{2}(\tau)$ is a modular form of weight $4 k+2$ over $\Gamma^{0}(2)$. Moreover,

$$
\begin{equation*}
P_{1}\left(-\frac{1}{\tau}\right)=2^{l} \tau^{4 k+2} P_{2}(\tau) \tag{17}
\end{equation*}
$$

On the other hand, following [5], write $\theta_{j}=\theta_{j}(0, \tau), 1 \leqslant j \leqslant 3$, and set $\delta_{1}(\tau)=\frac{1}{8}\left(\theta_{2}^{4}+\theta_{3}^{4}\right), \varepsilon_{1}(\tau)=\frac{1}{16} \theta_{2}^{4} \theta_{3}^{4}$, $\delta_{2}(\tau)=-\frac{1}{8}\left(\theta_{1}^{4}+\theta_{3}^{4}\right)$ and $\varepsilon_{2}(\tau)=\frac{1}{16} \theta_{1}^{4} \theta_{3}^{4}$. They admit Fourier expansion

$$
\begin{align*}
& \delta_{1}(\tau)=\frac{1}{4}+6 q+\cdots, \quad \varepsilon_{1}(\tau)=\frac{1}{16}-q+\cdots  \tag{18}\\
& \delta_{2}(\tau)=-\frac{1}{8}-3 q^{1 / 2}+\cdots, \quad \varepsilon_{2}(\tau)=q^{1 / 2}+\cdots \tag{19}
\end{align*}
$$

where the ". . ." terms are higher degree terms all having integral coefficients.
By [5, Lemma 2], we know that $\delta_{2}$ (resp. $\varepsilon_{2}$ ) is a modular form of weight 2 (resp. 4) over $\Gamma^{0}(2)$, and that $8 \delta_{2}, \varepsilon_{2}$ generate the ring of modular forms with integral coefficients over $\Gamma^{0}(2)$. Combining this argument with (7), (14) and (19), we obtain

$$
\begin{equation*}
P_{2}(\tau)=h_{0}\left(8 \delta_{2}\right)^{2 k+1}+h_{1}\left(8 \delta_{2}\right)^{2 k-1} \varepsilon_{2}+\cdots+h_{k}\left(8 \delta_{2}\right) \varepsilon_{2}^{k} \tag{20}
\end{equation*}
$$

where each $h_{r}, 0 \leqslant r \leqslant k$, is a canonically defined finite integral linear combination of the forms $\left\{\hat{A}\left(T M, \nabla^{T M}\right)\right.$ $\left.\operatorname{ch}\left(B_{j}, \nabla^{B_{j}}\right) \cosh \left(\frac{c}{2}\right)\right\}^{(8 k+4)}, j \geqslant 0$. For example, $h_{0}$ and $h_{1}$ can be written explicitly as $h_{0}=-\left\{\hat{A}\left(T M, \nabla^{T M}\right)\right.$ $\left.\cosh \left(\frac{c}{2}\right)\right\}^{(8 k+4)}$ and $h_{1}=\left\{\hat{A}\left(T M, \nabla^{T M}\right)\left[24(2 k+1)-\operatorname{ch}\left(B_{1}, \nabla^{B_{1}}\right)\right] \cosh \left(\frac{c}{2}\right)\right\}^{(8 k+4)}$.

Now recall that by [5, p. 36], $\delta_{i}, \varepsilon_{i}, i=1,2$, verify the transformation laws $\delta_{2}\left(-\frac{1}{\tau}\right)=\tau^{2} \delta_{1}(\tau), \varepsilon_{2}\left(-\frac{1}{\tau}\right)=$ $\tau^{4} \varepsilon_{1}(\tau)$. Using also (17) and (20), we find that

$$
\begin{equation*}
P_{1}(\tau)=2^{l}\left[h_{0}\left(8 \delta_{1}\right)^{2 k+1}+h_{1}\left(8 \delta_{1}\right)^{2 k-1} \varepsilon_{1}+\cdots+h_{k}\left(8 \delta_{1}\right) \varepsilon_{1}^{k}\right] . \tag{21}
\end{equation*}
$$

By (6), (13), (18), (21), and taking $q=0$, we get (8).

## Acknowledgements

This work was partially supported by MOEC and the 973 project. Part of the Note was written while the second author was visiting the Institute of Mathematics of Fudan University in February, 2003. He would like to thank Professors Xiaoman Chen and Jiaxing Hong for their hospitality.

## References

[1] L. Alvarez-Gaumé, E. Witten, Gravitational anomalies, Nucl. Phys. B 234 (1983) 269-330.
[2] M.F. Atiyah, F. Hirzebruch, Riemann-Roch theorems for differentiable manifolds, Bull. Amer. Math. Soc. 65 (1959) 276-281.
[3] K. Chandrasekharan, Elliptic Functions, Springer-Verlag, 1985.
[4] F. Han, W. Zhang, Modular invariance, characteristic numbers and $\eta$ invariants, Preprint, math-DG/0305289.
[5] K. Liu, Modular invariance and characteristic numbers, Comm. Math. Phys. 174 (1995) 29-42.
[6] K. Liu, W. Zhang, Elliptic genus and $\eta$-invariants, Internat. Math. Res. Notices 8 (1994) 319-328.
[7] S. Ochanine, Signature modulo 16, invariants de Kervaire géneralisé et nombre caractéristiques dans la $K$-théorie reelle, Mém. Soc. Math. France 109 (1987) 1-141.
[8] W. Zhang, Spin ${ }^{c}$-manifolds and Rokhlin congruences, C. R. Acad. Sci. Paris, Ser. I 317 (1993) 689-692.
[9] W. Zhang, Lectures on Chern-Weil Theory and Witten Deformations, in: Nankai Tracks in Math., Vol. 4, World Scientific, Singapore, 2001.


[^0]:    E-mail addresses: hanfeiycg@yahoo.com.cn (F. Han), weiping@nankai.edu.cn (W. Zhang).
    1631-073X/03/\$ - see front matter © 2003 Académie des sciences. Published by Éditions scientifiques et médicales Elsevier SAS. All rights reserved.
    doi:10.1016/S1631-073X(03)00241-3

