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Functional Analysis

A class of Banach spaces with no unconditional basic sequence

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Abstract

We give a construction of a reflexive Banach space X_{ω_1} with a transfinite basis of length ω_1 and with no unconditional basic sequence. In addition every bounded operator from a subspace of X_{ω_1} into the space X_{ω_1} is a sum of a simple diagonal operator and a strictly singular one. ***To cite this article: S.A. Argyros et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).***
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Résumé

Une classe d'espace de Banach sans suite basique inconditionnelle. Nous construisons un espace de Banach réflexif X_{ω_1} ayant une base transfinie de longueur ω_1 et n'admettant aucune suite basique inconditionnelle. De plus, tout opérateur borné d'un sous-espace de X_{ω_1} dans cet espace est somme d'un opérateur diagonal très simple et d'un opérateur strictement singulier. ***Pour citer cet article : S.A. Argyros et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).***
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Version française abrégée

Il est bien connu depuis plus de trente ans que tout espace de Banach X réflexif non séparable peut être décomposé en morceaux séparables par une longue suite de projections (voir [8]). Par conséquent, X admet des projections de norme 1 sur des sous-espaces séparables de dimension infinie. Cependant, il n'est pas du tout clair que l'algèbre de Banach engendrée par la suite de projections épuise, à petite perturbation près, l'algèbre de tous les opérateurs bornés sur X. Nous construisons un espace réflexif non séparable X_{ω_1} ayant cette propriété et qui est, en plus, purement non inconditionnel. Une propriété remarquable de X_{ω_1} est qu'il n'est isomorphe à aucun de ses hyperplans. De plus, dans l'esprit des travaux de Gowers [6], nous construisons une version inconditionnelle de X_{ω_1}, qui est un espace réflexif non séparable ayant une base inconditionnelle sur lequel aucun shift n'est borné, tout opérateur borné s'écrivant comme $D + S$, où D est diagonal et S est strictement singulier d'image séparable.

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Plus précisément, nous construisons une famille X_γ ($\gamma \leqslant \omega_1$) d'espaces de Banach réflexifs ayant une longue base transfinie, mais sans suite basique inconditionnelle. Pour décrire une telle suite d'espaces conditionnellement saturés et ayant beaucoup de projections il est nécessaire d'introduire des fonctionnelles dit « spéciales » qui imposeront cette conditionnalité. Cette construction, comme dans le cas de travail de Gowers et Maurey [7], dépend de façon cruciale d'un certain codage σ_ρ de suites finies de fonctionnelles. La différence essentielle avec le codage utilisé dans [7] est que σ_ρ n'est pas une injection, la propriété dont dépend les démonstrations de Gowers et Maurey. Nos démonstrations reposent sur une propriété « tree-like » du codage. Comme conséquence, sur chaque X_γ tout opérateur borné a une forme très simple.

Théorème 0.1. *Soit $T : X_\gamma \to X_\gamma$ un opérateur borné. Alors il existe un opérateur diagonal borné $D_T : X_\gamma \to X_\gamma$ tel que $T - D_T$ soit strictement singulier et tel que $D_T e_\alpha = D_T e_{\alpha+n}$, $\alpha < \gamma$ successeur.*

Comme les espaces X_γ ont des bases longues transfinies, l'algèbre $\mathcal{D}(X_\gamma)$ d'opérateurs de forme D_T est très riche. Dans beaucoup de cas nous identifions cette algèbre

Théorème 0.2. *$\mathcal{D}(X_{\omega^2})$ est isomorphe au dual de l'espace quasi-réflexif de James J_{T_0}, basé sur l'espace « mixed » de Tsirelson T_0 (cf.* [1]).

La propriété suivante est également remarquable.

Théorème 0.3. *Tout sous-espace de dimension finie de chaque X_γ peut être recopié par un $(4+\varepsilon)$-isomorphisme dans n'importe quelle région de tout autre X_δ.*

Nous donnons aussi pour chaque γ un sous-espace $X \hookrightarrow X_\gamma$ tel que l'espace $\mathcal{L}(X, X_\gamma)$ soit très riche, mais tout opérateur borné T de X dans X soit une variation strictement singulière d'un multiple de l'identité.

1. Introduction

We construct a family X_γ ($\gamma \leqslant \omega_1$) of reflexive Banach spaces with long transfinite bases but with no unconditional basic sequences. To describe such a family of conditionally saturated Banach spaces with many projections it was necessary to introduce a new construction of the so-called special functionals that impose this conditionality. In our spaces X_γ every bounded operator T is split into its diagonal part D_T and its strictly singular part S_T. Having long transfinite bases our spaces X_γ have rich spaces $\mathcal{D}(X_\gamma)$ of diagonal operators. Rather unexpectedly, in many cases we were able to identify the spaces $\mathcal{D}(X_\gamma)$. For example, we show that $\mathcal{D}(X_{\omega^2})$ is isomorphic to the dual of the quasi-reflexive Banach space J_{T_0}, the James space over the mixed Tsirelson space T_0 (see [1]). We discover several other new phenomena by looking at subspaces X of our spaces X_γ. For example, we show that every finite-dimensional subspace of any X_γ can be moved by an $(4+\varepsilon)$-isomorphism to essentially any region of any other member X_δ of our class. We also find subspaces X of X_γ such that the operator space $\mathcal{L}(X, X_\gamma)$ is quite rich but any bounded operator T from X into X is a strictly singular variation of a scalar multiple of the identity.

2. The norming set

The maximal space in our class X_{ω_1} will be defined as the completion of $(c_{00}(\omega_1), \|\cdot\|_{\omega_1})$ under the norm $\|\cdot\|_{\omega_1}$ induced by a set of functionals $K_{\omega_1} \subseteq c_{00}(\omega_1)$.

2.1. Definition of K_{ω_1}

Let $(m_j)_j$, $(n_j)_j$ be the two sequences defined recursively as follows: $m_1 = 2$, $n_1 = 4$, $m_{j+1} = m_j^5$, and $n_{j+1} = (5n_j)^{s_j}$ for $s_j = \log_2(m_{j+1})$. The set K_{ω_1} is the minimal subset of $c_{00}(\omega_1)$ satisfying that: (1) It contains $(e^*_\gamma)_{\gamma<\omega_1}$, is symmetric and is closed under the restriction on intervals of ω_1. (2) $\phi = \frac{1}{m_{2j}} \sum_{i=1}^{n_{2j}} \phi_i \in K_{\omega_1}$ for $\{\phi_i\colon\ i = 1, \ldots, n_{2j}\} \subseteq K_{\omega_1}$ with $\operatorname{supp}\phi_1 < \cdots < \operatorname{supp}\phi_{n_{2j}}$, and we say that ϕ is a result of a $(1/m_{2j}, \mathcal{A}_{n_{2j}})$

operation. (3) $\phi = \frac{1}{m_{2j+1}} \sum_{i=1}^{n_{2j+1}} \phi_i \in K_{\omega_1}$ for special sequences $(\phi_1, \ldots, \phi_{n_{2j+1}}) \subseteq K_{\omega_1}$ (see Subsection 2.3) and we say that ϕ is a result of a $(1/m_{2j+1}, \mathcal{A}_{n_{2j+1}})$ operation. (4) It is rationally convex. The norm on $c_{00}(\omega_1)$ is defined as $\|x\|_{\omega_1} = \sup\{\phi(x) = \sum \phi(\alpha) \cdot x(\alpha)\colon\ \phi \in K_{\omega_1}\}$ and X_{ω_1} is the completion of $(c_{00}(\omega_1), \|\cdot\|_{\omega_1})$. Each of the above four properties provides certain features in the space X_{ω_1}. The first makes the family $(e_\gamma)_{\gamma<\omega_1}$ a transfinite bimonotone basis of X_{ω_1}. The second saturates X_{ω_1} with local unconditional structure. This property will be responsible for the existence of seminormalized averages in every block sequence of X_{ω_1}. The third property saturates X_{ω_1} with conditional structure and will make impossible for X_{ω_1} to contain any unconditional basic sequence. Finally, the fourth property is a tool for proving properties of the space of operators from an arbitrary subspace X of X_{ω_1} into X_{ω_1}. The above definition, with the exception of the missing definition of special sequences, is based on the corresponding definitions from [2] and [3] which in turn are variants of the construction from [7]. By the minimality of K_{ω_1} each $\phi \in K_{\omega_1}$ has one of the following forms: (i) ϕ is of *type* 0 if $\phi = \pm e_a^*$. (ii) ϕ is of *type* I if $\phi = \pm Ef$ for f a result of one $(1/m_j, \mathcal{A}_{n_j})$ operation and E an interval. (iii) ϕ is of *type* II if ϕ is a rational convex combination of type 0 and type I functionals.

Definition 2.1. For $\phi \in K_{\omega_1}$ of type I we define the *weight* of ϕ, denoted by $w(f)$, to be m_j if ϕ is the result of an $(\frac{1}{m_j}, \mathcal{A}_{n_j})$ operation. Notice that $\phi \in K_{\omega_1}$ may have many weights.

The definition of the special sequences will, as in the case [7], depend crucially on certain coding σ_ρ. The essential difference is that now σ_ρ is not an injection, a crucial property on which the proofs in [7] rely. Our proofs on the other hand will rely on a 'tree-like property' of our coding which we now describe. First we notice that each $2j+1$-special sequence $\Phi = (\phi_1, \phi_2, \ldots, \phi_{n_{2j+1}})$ is of the form $\operatorname{supp}\phi_1 < \cdots < \operatorname{supp}\phi_{n_{2j+1}}$ with each ϕ_i of type I. The *tree-like property* is the following: For any pair of $2j+1$-special sequences $\Phi = (\phi_1, \phi_2, \ldots, \phi_{n_{2j+1}})$, $\Psi = (\psi_1, \psi_2, \ldots, \psi_{n_{2j+1}})$ there exist $1 \leqslant k_{\Phi,\Psi} \leqslant \lambda_{\Phi,\Psi} \leqslant n_{2j+1}$ such that (i) If $1 \leqslant k < k_{\Phi,\Psi}$ then $\phi_k = \psi_k$ and if $k_{\Phi,\Psi} < k < \lambda_{\Phi,\Psi}$ then $w(\phi_k) = w(\psi_k)$. (ii) $(\bigcup_{k_{\Phi,\Psi}<k<\lambda_{\Phi,\Psi}} \operatorname{supp}\phi_k) \cap (\bigcup_{k_{\Phi,\Psi}<k<\lambda_{\Phi,\Psi}} \operatorname{supp}\psi_k) = \emptyset$. (iii) $\{w(\phi_k)\colon\ \lambda_{\Phi,\Psi} < k < n_{2j+1}\} \cap \{w(\psi_k)\colon\ \lambda_{\Phi,\Psi} < k < n_{2j+1}\} = \emptyset$.

Comparing the above tree-like property with the corresponding property from [7], we notice that the new ingredient is the number $\lambda_{\Phi,\Psi}$. Its occurrence is a byproduct of the fact that the coding σ_ρ is not one-to-one. The property (ii) will however give us a sufficient control of our special functionals.

2.2. ρ-functions

By a ρ-function we understand a function $\rho\colon [\omega_1]^2 \to \omega$ such that for all $\alpha < \beta < \gamma$ and $n \in \omega$: (a) $\rho(\alpha,\gamma) \leqslant \max\{\rho(\alpha,\beta), \rho(\beta,\gamma)\}$ and $\rho(\alpha,\beta) \leqslant \max\{\rho(\alpha,\gamma), \rho(\beta,\gamma)\}$. (b) The set $\{\alpha < \beta\colon\ \rho(\alpha,\beta) \leqslant n\}$ is finite.

The ρ-functions have been introduced in [10] (see also [11]). The following will related ρ-functions to our coding.

Definition 2.2. Let F be a finite subset of ω_1. We set $n_F = \max\{\rho(a,\beta)\colon\ a,\beta \in F\}$. For $n \in \omega$ with $n_F \leqslant n$ and $\gamma = \max F$ we define the *n-completion* of F to be the set $\operatorname{Cl}(F,n) = \{\alpha\colon\ \alpha \leqslant \gamma \text{ and } \rho(\alpha,\gamma) \leqslant n\}$ (convention: $\rho(\gamma,\gamma) = 0$ for all γ). The condition (b) of the definition of ρ-function yields that $\operatorname{Cl}(F,n)$ is finite. A finite subset G of ω_1 is said to be *n-complete* if $\operatorname{Cl}(G,n) = G$.

Basic properties from the definition are that if F, G are two finite n-complete subsets of ω_1, then for each $\gamma \in F$ the set $F_\gamma = \{\beta\colon\ \beta \in F,\ \beta \leqslant \gamma\}$ is also n-complete, and that if $F \cap G \neq \emptyset$ then $F \cap G$ defines an initial part of both F and G.

2.3. The σ_ρ-coding and the special sequences

We denote by $\mathbb{Q}_s(\omega_1)$ the following set: $\mathbb{Q}_s(\omega_1) = \{(\phi_1, M_1, \phi_2, M_2, \ldots, \phi_d, M_d, N)\colon\ 0 < d \in \omega, \phi_k \in c_{00}(\omega_1, \mathbb{Q}),\ \operatorname{supp}\phi_1 < \operatorname{supp}\phi_2 < \cdots < \operatorname{supp}\phi_d,\ M_1, M_2, \ldots, M_d < N \in \omega,\ N \geqslant n_{\cup_{k=1}^d \operatorname{supp}\phi_k}\}$ and by $\mathbb{Q}_s$ the corresponding set when each $\phi_k \in c_{00}(\omega, \mathbb{Q})$.

To define σ_ρ we start with a one-to-one function $\sigma:\mathbb{Q}_s\to\{2j\colon\ j\in\mathbb{N}\}$ such that $\sigma(\phi_1,M_1,\phi_2,M_2,\dots,\phi_d,M_d,N)>\max\{N^2,\frac{1}{\varepsilon^2},\max\operatorname{supp}\phi_d\}$, where $\varepsilon=\min\{|\phi_k(e_\alpha)|\colon\ \alpha\in\operatorname{supp}\phi_k,\ k=1,\dots,d\}$. Given a finite subset F of ω_1, let $\pi_F:\{1,2,\dots,\#F\}\to F$ be the order preserving map. For $\Phi=(\phi_1,M_1,\phi_2,M_2,\dots,\phi_d,M_d,N)\in\mathbb{Q}_s(\omega_1)$ we denote by $G_\Phi=\mathrm{Cl}(\bigcup_{i=1}^d\operatorname{supp}\phi_i,N)$ and then we consider the family $\tilde{\Phi}=(\tilde{\phi}_1,M_1,\tilde{\phi}_2,M_2,\dots,\tilde{\phi}_d,M_d,N)\in\mathbb{Q}_s$, where $\tilde{\phi}_k=\phi_k\circ\pi_{G_\Phi}$. Finally, we set $\sigma_\rho(\Phi)=\sigma(\tilde{\Phi})$.

A sequence $(\phi_i)_{i=1}^{n_{2j+1}}$ of functionals of K_{ω_1} is said be $2j+1$ *special sequence* if: (i) $\operatorname{supp}\phi_1<\operatorname{supp}\phi_2<\cdots<\operatorname{supp}\phi_{n_{2j+1}}$, each ϕ_k is of type I, $w(\phi_k)=m_{2j_k}$ and $w(\phi_1)\geqslant m_{2j+1}$. (ii) For each $1\leqslant k<n_{2j+1}$ there exists $N_k\in\omega$ such that $w(\phi_{k+1})=m_{\sigma_\rho(\Phi_k)}$ for $\Phi_k=(\phi_1,w(\phi_1),\phi_2,w(\phi_2),\dots,\phi_k,w(\phi_k),N_k)$.

3. The structure of X_{ω_1} and its subspaces

As we have mentioned before, the main property of X_{ω_1} is that it does not contain any unconditional basic sequence. A more detailed description of the properties of X_{ω_1} and its subspaces follows.

Proposition 3.1. (i) *The space X_{ω_1} is reflexive.*

(ii) *The closed linear span of a block sequence of X_{ω_1} is hereditarily indecomposable, and hence every two nonseparable subspaces of X_{ω_1} satisfy the zero angle property.*

(iii) *For I_1, I_2 infinite disjoint intervals the spaces X_{I_1} and X_{I_2} are totally incomparable.*

(iv) *Any projection $P_Y:X_{\omega_1}\to Y$ has the form $P=\sum_{i=1}^n P_{X_{I_i}}+S$ for S strictly singular, $\{I_i\}_i$ intervals, and X_{ω_1} is not isomorphic to any of its proper subspaces.*

Two additional properties of the ρ-function yield more properties of $(X_\gamma)_{\gamma\leqslant\omega_1}$.

Definition 3.2. A ρ-function is said to be *smooth* if for each limit ordinal $\gamma<\omega_1$ there exists $C_\gamma>0$ such that $\#\{\beta\colon\ \rho(\beta,\gamma)\leqslant n\}\leqslant C_\gamma\cdot n$ for all $n\in\omega$. Roughly speaking, a ρ-function is called *universal* if it has the property that its restriction on any infinite interval of ω_1 contains all the possible elementary models of ρ-functions. Functions with both properties exist.

Proposition 3.3. *If the coding σ_ρ is based on a smooth ρ-function then for each $\gamma<\omega_1$ the transfinite basis $(e_\beta)_{\beta<\gamma}$ can be reordered as $(e_{\beta_n})_n$ to become a Schauder basis for X_γ.*

Definition 3.4. A transfinite basis $(e_\gamma)_{\gamma<\delta}$ is said to be *nearly subsymmetric* if there exists $C>0$ such that for every family of finite successive subsets $\{F_i\}_{i=1}^d$ of δ and every family $\{I_i\}_{i=1}^d$ of successive infinite intervals there exists $\{G_i\}_{i=1}^d$ with $G_i\subseteq I_i$, $\#G_i=\#F_i$ and the order preserving map $T:\langle(e_\alpha)_{\alpha\in\cup_{i=1}^d F_i}\rangle\to\langle(e_\beta)_{\beta\in\cup_{i=1}^d G_i}\rangle$ satisfies $\|T\|\cdot\|T^{-1}\|\leqslant C$.

Proposition 3.5. *If the coding σ_ρ is based on a universal ρ-function then the basis $(e_\gamma)_{\gamma<\omega_1}$ is nearly subsymmetric.*

Corollary 3.6. *There is an H.I.-space with a nearly subsymmetric Schauder basis.*

Let us say that two basic sequences are *finitely equivalent* if there exists $C>0$ such that any finite subset of each one of them is C-equivalent to a finite subset of the other through the natural order preserving operator. Then the family $\{X_{(\omega\gamma,\omega\gamma+\omega)}\}_{\gamma<\omega_1}$ is an example of an uncountable family of reflexive HI spaces that are pairwise totally incomparable though their bases are pairwise finitely equivalent.

4. The operator spaces

We proceed with some results regarding the structure of the space $\mathcal{L}(X_{\omega_1})$ including the structure of $\mathcal{L}(Y)$ with Y a subspace of X_{ω_1}. We say that $\xi:\omega_1\to\mathbb{R}$ is a *step function* provided that $\alpha\leqslant\beta<\alpha+\omega$ implies $\xi(\beta)=\xi(\alpha+\omega)$. For a given step function $\xi:\omega_1\to\mathbb{R}$, the corresponding diagonal operator X_{ω_1} is defined by $D_\xi(e_\alpha)=\xi(\alpha)e_\alpha$. We denote by $\mathcal{D}(X_{\omega_1})$ the closed commutative subalgebra of $\mathcal{L}(X_{\omega_1})$ consisting of all such diagonal operators. Similarly for $\gamma\leqslant\omega_1$ we denote by $\mathcal{D}(X_\gamma)$ the corresponding subspace of $\mathcal{L}(X_\gamma)$. Finally we extend this notation to the subspaces Y of X_{ω_1} generated by a normalized block sequence $(x_\beta)_{\beta<\gamma}$. For any space Y we denote by $\mathcal{S}(Y)$ the ideal of strictly singular operators, and $\mathcal{L}(Y,X_{\omega_1})$, $\mathcal{S}(Y,X_{\omega_1})$ have the analogous meaning.

Theorem 4.1. *For every bounded operator $T:X_{\omega_1}\to X_{\omega_1}$ there is a continuous step function $\xi_T:\omega_1\to\mathbb{R}$ and a strictly singular operator $S_T:X_{\omega_1}\to X_{\omega_1}$ such that $T=D_{\xi_T}\oplus S_T$. Moreover* (a) *There exists $C>0$ such that $\|D_{\xi_T}\|\leqslant C\|T\|$. Hence $\mathcal{L}(X_{\omega_1})\cong\mathcal{D}(X_{\omega_1})\oplus\mathcal{S}(X_{\omega_1})$.* (b) *For every $S\in\mathcal{S}(X_{\omega_1})$ the transfinite sequence $(\|Se_\alpha\|)_{\alpha<\omega_1}$ belongs to $c_0(\omega_1)$.*

The corresponding results for the structure of $\mathcal{L}(X_\gamma)$, $\gamma<\omega_1$, are similarly obtained. Namely $\mathcal{L}(X_\gamma)\cong\mathcal{D}(X_\gamma)\oplus\mathcal{S}(X_\gamma)$. For understanding the norm of the diagonal operators we use the well-known mixed Tsirelson space $T_0=\overline{\langle c_{00},\|\cdot\|_{T_0}\rangle}$, where $\|x\|_{T_0}=\max\{\|x\|_0,\sup_j\sup\frac{1}{m_{2j}}\sum_{k=1}^{n_{2j}}\|E_kx\|_{T_0}\}$ and where the supremum is taken over all families $\{E_k\}_{k=1}^n$ of successive subsets of ω. As in the case of Schlumprecht space [9] T_0 is reflexive with a subsymmetric basis. Next, for any infinite set of ordinals A we define the space $J_{T_0}(A)$ to be the completion of $(c_{00}(A),\|\cdot\|_{J_{T_0}})$, where $\|x\|_{J_{T_0}}=\sup\{\|\sum_{k=1}^n(\sum_{\alpha\in I_k}x(\alpha))e_k\|_{T_0}\colon\{I_k\}_{k=1}^n$ a family of disjoint successive intervals of $A\}$.

The definition of $J_{T_0}(A)$ is the natural fusion of the notion of 'Jamesfication' of a given Banach space introduced in [4] and the definition of the long James spaces as defined by G. Edgar [5]. In particular, for $A=\omega$ the space $J_{T_0}=J_{T_0}(\omega)$ is quasi-reflexive. The following is our main result regarding $\mathcal{D}(X_\gamma)$.

Theorem 4.2. *For each infinite $\gamma\leqslant\omega_1$ there is some $A\subseteq\gamma$ such that $\mathcal{D}(X_\gamma)\cong J_{T_0}^*(A)$.*

It follows that for a countable ordinal γ, the space $\mathcal{D}(X_\gamma)$ is separable (since $\dim J_{T_0}^*(A)=\#A$) and it does not contain c_0, and as a particular case $\mathcal{D}(X_{\omega^2})\cong J_{T_0}^*$. The previous result is extended for operators acting on subspaces of X_{ω_1}.

Theorem 4.3. *Let $X\hookrightarrow X_{\omega_1}$. Then for some $\gamma\leqslant\omega_1$ $\mathcal{L}(X,X_{\omega_1})\cong\mathcal{D}(X_\gamma,X_{\omega_1})\oplus\mathcal{S}(X,X_{\omega_1})$. If in addition X is generated by a block sequence, then $\mathcal{L}(X)\cong\mathcal{D}(X)\oplus\mathcal{S}(X)$ and $\mathcal{D}(X)\cong J_{T_0}^*(A)$.*

However, for every $\gamma<\omega_1$ there exists a subspace $Y_\gamma\hookrightarrow X_{\omega_1}$ such that $\mathcal{L}(Y_\gamma)\cong\langle Id_{Y_\gamma}\rangle\oplus\mathcal{S}(Y_\gamma)$ while $\mathcal{L}(Y_\gamma,X_{\omega_1})\cong\mathcal{D}(X_\gamma,X_{\omega_1})\oplus\mathcal{S}(Y_\gamma,X_{\omega_1})$.

We now explain the structural property of X_{ω_1} that yields the isomorphism between $\mathcal{D}(X_\gamma)$ and $J_{T_0}^*(A)$.

Definition 4.4. Let X,Y be Banach spaces with a transfinite basis $(x_\gamma)_{\gamma<\delta}$ and a Schauder basis $(y_n)_n$ respectively. We say that Y is *finitely interval representable* in X (*f.i.r.* in short) if there exists $C>0$ such that for every $n<\omega$ and $I_1\leqslant\cdots\leqslant I_n$ not necessarily distinct intervals of δ there exists a block sequence $(z_l)_{l=1}^n$ with $z_l\in\langle x_\alpha\rangle_{\alpha\in I_l}$ such that the natural mapping $T:\langle(y_l)_{l=1}^n\rangle\to\langle(z_l)_{l=1}^n\rangle$ satisfies $\|T\|\cdot\|T^{-1}\|<C$.

The main reason for $\mathcal{D}(X_\gamma)\cong J_{T_0}^*(A)$ is contained in the following fact.

Theorem 4.5. *J_{T_0} is f.i.r. in the closed span of any transfinite block sequence of X_{ω_1}.*

Notice that since the basis of J_{T_0} is not unconditional, the representability of J_{T_0} derived by the previous Theorem essentially occurs only in the case of H.I.-saturated Banach spaces. We note that the relation between the representability of J_{T_0} in X_{ω_1} and the structure of $\mathcal{D}(X_\alpha)$ is a consequence of a minimality property described by the following result.

Proposition 4.6. *Suppose Y has a basis $(y_n)_n$ which dominates the summing basis of c_0. If Y is f.i.r. in X_{ω_1} then $(y_n)_n$ also dominates the basis of J_{T_0}.*

We also notice that the space T_0 satisfies the corresponding minimality property among all Y with an unconditional basis $(y_n)_n$ which are finitely interval representable in X_{ω_1}.

5. The unconditional counterpart

The space X_{u,ω_1} is the counterpart of X_{ω_1} in the frame of the spaces with an unconditional basis, similar to [6]. The space is defined as for X_{ω_1} by a norming family of functionals K_{u,ω_1} satisfying (1)–(4) and:

(5) It is closed under the restriction of all functionals with odd weight to every subset of ω_1.

Although X_{u,ω_1} belongs to the class of the spaces with an unconditional basis its study uses the same tools used in the study of X_{ω_1}. For example, given a bounded operator $T : X_{u,\omega_1} \to X_{u,\omega_1}$ the transfinite sequence $(\operatorname{dist}(Te_\gamma, \mathbb{R}e_\gamma))_{\gamma<\omega_1}$ belongs to $c_0(\omega_1)$, and the operator T is strictly singular if and only if the sequence $(\|Te_\gamma\|)_{\gamma<\omega_1}$ belongs to $c_0(\omega_1)$.

Theorem 5.1. *The space X_{u,ω_1} is nonseparable reflexive with an unconditional basis and $\mathcal{L}(X_{u,\omega_1}) \cong \mathcal{D}(X_{u,\omega_1}) \oplus \mathcal{S}(X_{u,\omega_1})$.*

Here $\mathcal{D}(X_{u,\omega_1})$ denotes the space of the diagonal operators and $\mathcal{S}(X_{u,\omega_1})$ is the space of strictly singular operators S with separable range such that $e^*_\gamma(Se_\gamma) = 0$ for every $\gamma < \omega_1$.

Corollary 5.2. *There is a non-separable reflexive space X with an unconditional basis such that* (a) *X is not isomorphic to any of its proper subspaces.* (b) *Every bounded linear operator $T : X \to X$ is of the form $D + S$ with D a diagonal operator and S a strictly singular operator with separable range.* (c) *For every I_1, I_2 infinite disjoint subsets of ω_1 the spaces X_{I_1}, X_{I_2} are totally incomparable.*

Moreover if we consider the version of X_{u,ω_1} obtained by a universal ρ-function then the unconditional basis $(e_\gamma)_{\gamma<\omega_1}$ is nearly subsymmetric and for any pair of disjoint infinite intervals I_1, I_2 the unconditional basic sequences $(e_\alpha)_{\alpha\in I_1}$, $(e_\alpha)_{\alpha\in I_2}$ are finitely equivalent while the corresponding spaces X_{I_1}, X_{I_2} are totally incomparable.

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