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Semimartingale attractors for generalized Allen–Cahn SPDEs driven by space–time white noise

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Abstract

We introduce the notion of a semimartingale attractor associated with space–time white noise driven generalized Allen–Cahn SPDEs. We treat the driving noise in the martingale measure setting, and we give an existence result for this type of random attractors for these generalized Allen–Cahn SPDEs in our setting. This Note focuses on semimartingale functional attractors, but our noise setting also leads naturally to a related type of random attractors that we call semimartingale measure attractors and which we detail in an upcoming article. Detailed proofs and extensions of our result, as well as other properties of semimartingale attractors, for different types of SPDEs are also furnished in the follow-up article. ***To cite this article: H. Allouba, J.A. Langa, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Attracteurs semimartingales pour des EDPS d'Allen–Cahn généralisées dirigées par le bruit blanc spatio–temporel. Nous introduisons la notion d'attracteur semimartingale associé aux EDPS d'Allen–Cahn généralisées dirigées par un bruit blanc spatio–temporel. Nous traitons notre bruit dans le cadre de mesure martingale, et nous donnons un résultat d'existence pour ce type d'attracteurs pour ces EDPS d'Allen–Cahn généralisées. Cette Note discute des attracteurs semimartingales fonctionnels, mais notre cadre du bruit mene aussi naturellement à un type lié d'attracteurs aléatoires que nous appelons attracteurs de mesure semimartingale et que nous détaillerons dans un prochain article. En outre, les preuves détaillées, avec d'autres propriétés d'attracteurs semimartingales et des extensions de notre resultat appliquées à différents types d'EDPS, seront aussi fournies dans ce prochain article. ***Pour citer cet article : H. Allouba, J.A. Langa, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Version française abrégée

Soient $\mathcal{C}_L \triangleq \mathbb{R}_+ \times [0,L]$, $\mathcal{O}_L \triangleq (0,\infty) \times (0,L)$ et $\partial^2 W(t,x)/\partial t\partial x$ le bruit blanc spatio–temporel. Nous considérons l'EDPS :

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$$\begin{cases} \dfrac{\partial U}{\partial t} = \Delta_x U + f(U) + \dfrac{\partial^2 W}{\partial t \partial x}, & (t,x) \in \mathcal{O}_L, \\ U(t,0) = U(t,L) = 0, & 0 < t < \infty, \\ U(0,x) = u_0(x), & 0 \leqslant x \leqslant L, \end{cases} \tag{1}$$

où $f : \mathbb{R} \to \mathbb{R}$ a la forme :

$$f(u) = \sum_{k=0}^{2p-1} a_k u^k, \quad \text{avec } p \in \mathbb{N}, \text{ et } a_{2p-1} < 0 \tag{2}$$

(e.g., f est une fonction de type Allen–Cahn $f(u) = 2u(1-u^2)$), et $u_0 \in C([0,L];\mathbb{R})$ est une fonction déterministe. Nous introduisons la notion d'attracteur semimartingale fonctionnel associé à l'EDPS (1) dirigée par un bruit blanc spatio–temporel, et nous donnons un résultat d'existence pour cet attracteur aléatoire. Le bruit blanc spatio–temporel est traité ici comme une mesure martingale orthogonale continue, désignée par $\mathcal{W}$, et liée au drap brownien W comme dans Walsh [12]. Il est important de noter que notre cadre de mesure martingale spatio–temporel pour le bruit et notre notion d'attracteur semimartingale sont nouveaux dans la littérature des attracteurs aléatoires. Ce cadre peut également servir dans l'analyse stochastique des attracteurs aléatoires, et il nous permet d'introduire et de traiter la nouvelle notion des attracteurs de mesure semimartingale [3]. Ce dernier type d'attracteurs est basé sur la notion de mesures semimartingales qui généralise la notion de mesures semimartingale orthogonales continues introduite dans [2]. L'article [3] contient aussi la preuve détaillée de notre résultat présenté ici. En outre, quelques extensions et propriétés d'attracteurs semimartingales pour différents types d'EDPS sont aussi fournies dans [3].

Nous remarquons que Crauel et Flandoli [6] et Schmalfuss [10] ont introduit le concept d'attracteur pour quelques EDPS, et ceci a été utilisé avec succès dans l'étude de quelques propriétés qualitatives pour ces équations (voir Schmalfuss [10], Crauel et al. [7], et Caraballo et al. [5]).

1. Statements and discussions of results

We consider the Allen–Cahn type SPDE (1) on $\mathcal{C}_L \triangleq \mathbb{R}_+ \times [0,L]$, where $L > 0$ is fixed, and $W(t,x)$ is the Brownian sheet corresponding to the driving space–time white noise, written formally as $\partial^2 W/\partial t \partial x$. As in Walsh [12], white noise is regarded as a continuous orthogonal martingale measure, which we denote by $\mathcal{W}$. We assume that f satisfies the conditions in (2) (in particular f can be the celebrated Allen–Cahn term $f(u) = 2u(1-u^2)$ as in [1]). We assume that the initial function $u_0 \in C([0,L];\mathbb{R})$ to be deterministic. Henceforth, we will denote (1) by $e_{AC}(f,u_0)$. We introduce the concept of a semimartingale functional attractor corresponding to a space–time white noise driven SPDE, and we show the existence of such a random attractor for $e_{AC}(f,u_0)$ in our noise setting. It is important to note that our martingale measure noise setting is new in the random attractors literature. It paves the way for new stochastic analytic investigations of random attractors, and it is also crucial in our introduction and treatment of a related new type of random attractors which we call semimartingale measure attractors (more on that in [3]). This latter type of random attractors is based on the notion of semimartingale measures which generalizes the concept of continuous orthogonal semimartingale measures introduced in [2].

Crauel and Flandoli [6] and Schmalfuss [10] have introduced the concept of an attractor for some SPDEs, and this has been successfully used in the study of qualitative properties for these equations (see, among others, Schmalfuss [10], Crauel et al. [7], and Caraballo et al. [5]). This concept has been developed within the framework of the theory of random dynamical systems (Arnold [4]).

Proceeding toward a precise statement of our results, let us recall some definitions associated with random attractors. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\theta_t : \Omega \to \Omega,\ t \in \mathbb{R}\}$ a family of measure preserving transformations such that $(t,\omega) \mapsto \theta_t\omega$ is measurable, $\theta_0 = \mathrm{id}$, $\theta_{t+s} = \theta_t\theta_s$, for all $s,t \in \mathbb{R}$. The flow θ_t together with the probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is called a *measurable dynamical system*. Furthermore, we suppose that the shift θ_t is ergodic. A *random dynamical system* (RDS) on a complete metric (or Banach) space $(\mathbb{B}, d)$ with Borel σ-algebra $\mathcal{B}$, over θ on $(\Omega, \mathcal{F}, \mathbb{P})$ is a measurable map $\mathbb{R}_+ \times \Omega \times \mathbb{B} \ni (t, \omega, \xi) \mapsto \Phi(t, \omega)\xi \in \mathbb{B}$ such that $\mathbb{P}$-a.s.

(i) $\Phi(0, \omega) = \mathrm{id}$ (on $\mathbb{B}$),
(ii) $\Phi(t+s, \omega) = \Phi(t, \theta_s\omega) \circ \Phi(s, \omega)$, $\forall t, s \in \mathbb{R}_+$ (cocycle property).

A RDS is continuous (differentiable) if $\Phi(t, \omega): \mathbb{B} \to \mathbb{B}$ is continuous (differentiable). A random set $K(\omega) \subset \mathbb{B}$ is said to *absorb* the set $D \subset \mathbb{B}$ if there exists a random time $t_D(\omega)$ such that

$$t \geqslant t_D(\omega) \Rightarrow \Phi(t, \theta_{-t}\omega)D \subset K(\omega), \text{ a.s. } \mathbb{P}.$$

$K(\omega)$ is forward invariant if $\Phi(t, \omega)K(\omega) \subseteq K(\theta_t\omega)$, for all $t \in \mathbb{R}_+$ a.s. $\mathbb{P}$. Now, let $\mathrm{dist}(\cdot, \cdot)$ denote the Hausdorff semidistance $\mathrm{dist}(B_1, B_2) = \sup_{\xi_1 \in B_1} \inf_{\xi_2 \in B_2} d(\xi_1, \xi_2)$, for $B_1, B_2 \subset \mathbb{B}$. A random set $\mathcal{A}_\Phi(\omega) \subset \mathbb{B}$ is said to be a *random attractor* associated with the RDS Φ if $\mathbb{P}$-a.s.

(i) $\mathcal{A}_\Phi(\omega)$ is compact and, for all $\xi \in \mathbb{B}$, the map $\xi \mapsto \mathrm{dist}(\xi, \mathcal{A}_\Phi(\omega))$ is measurable,
(ii) $\Phi(t, \omega)\mathcal{A}_\Phi(\omega) = \mathcal{A}_\Phi(\theta_t\omega)$, $\forall t \geqslant 0$ (invariance), and
(iii) for all $D \subset \mathbb{B}$ bounded (and nonrandom) $\lim_{t\to\infty} \mathrm{dist}(\Phi(t, \theta_{-t}\omega)D, \mathcal{A}_\Phi(\omega)) = 0$.

Remark 1. Note that $\Phi(t, \theta_{-t}\omega)\xi$ can be interpreted as the position at $t = 0$ of the trajectory which was in ξ at time $-t$. Thus, the attraction property holds from $t = -\infty$.

We now set the stage for our result. Let U be the solution to $e_{AC}(f, u_0)$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ (see Appendix), and let $U^\varphi(t) \triangleq (U(t) - u_0, \varphi)$; $\forall \varphi \in \Theta_0^L$. Then, by the assumptions on f we have easily that $\{U^\varphi(t);\ t \in \mathbb{R}_+\}$ is a semimartingale on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ for each φ since (TFF) (see Appendix) gives

$$U^\varphi(t) = \int_0^t \big(U(s), \varphi''\big)\,\mathrm{d}s + \int_0^t \big(f\big(U(s), \varphi\big)\big)\,\mathrm{d}s + \int_0^L\!\!\int_0^t \varphi(x)\mathcal{W}(\mathrm{d}s, \mathrm{d}x); \quad 0 \leqslant t < \infty, \text{ a.s. } \mathbb{P}. \tag{3}$$

We call the random field $U(t, x)$ a weak semimartigale sheet. Now, set $\Phi_U(t-s, \omega)U(s, \cdot, \omega) = U(t, \cdot, \omega)$. We call a random attractor $\mathcal{A}_{\Phi_U}$ associated with Φ_U a semimartingale functional attractor. When we want to emphasize it, we say weak semimartingale functional attractor. Our result for $\mathcal{A}_{\Phi_U}$ can now be stated as

Theorem 1.1. *Suppose f satisfies* (2) *and $u_0: [0, L] \to \mathbb{R}$ is continuous and deterministic. Then, the SPDE $e_{AC}(f, u_0)$ has a strong pathwise-unique solution and it possesses a semimartingale functional attractor $\mathcal{A}_{\Phi_U} \subset L^2(0, L)$.*

Remark 2. It is well known [12] that the two formulations (GFF) and (TFF) are equivalent if the drift and diffusion coefficients are locally Lipschitz, which they clearly are for $e_{AC}(f, u_0)$. Also, we will use the notation $\mathcal{R}_{t,L}$ for $[0, t] \times [0, L]$.

2. Sketch of proof of Theorem 1.1

Let $\beta \geqslant 0$; let $Z_\beta(t, x)$ be the pathwise-unique strong solution to (1) with $f(Z_\beta) = -\beta Z_\beta$ and $Z_\beta(0, x) \equiv 0$, which is Hölder continuous with $\alpha_{\text{time}} = 1/4 - \varepsilon$ in time and $\alpha_{\text{space}} = 1/2 - \varepsilon$ in space, $\forall \varepsilon > 0$ (a standard result as in [12], pp. 321–322); and consider the equation

$$
\begin{aligned}
V_\beta(t,x) &= \int_0^L G_t(x,y)u_0(y)\,\mathrm{d}y + \int_0^L\int_0^t \big[f(V_\beta+Z_\beta)(s,y)+\beta Z_\beta(s,y)\big]G_{t-s}(x,y)\,\mathrm{d}s\,\mathrm{d}y \\
&\triangleq \int_0^L G_t(x,y)u_0(y)\,\mathrm{d}y + I_\beta(t,x) = H(t,x) + I_\beta(t,x),
\end{aligned}
\tag{4}
$$

i.e., the random PDE:

$$
\begin{cases}
\dfrac{\partial V_\beta}{\partial t} = \Delta_x V_\beta + f(V_\beta+Z_\beta) + \beta Z_\beta, & (t,x)\in \mathcal{O}_L,\\
V_\beta(t,0) = V_\beta(t,L) = 0, & 0<t<\infty,\\
V_\beta(0,x) = u_0(x), & x\in[0,L].
\end{cases}
\tag{5}
$$

We first briefly sketch the proof of the existence–uniqueness for (1). We note that when $p=1$ in (2) f is linear, and the strong existence and pathwise uniqueness for $e_{AC}(f,u_0)$ (see Appendix for definitions) follow as in [12], pp. 321–322. We now turn to the case $p>1$. Let $\beta=0$, and let $Z\triangleq Z_0$ and $V\triangleq V_0$. Clearly, the existence and uniqueness for $e_{AC}(f,u_0)$ is equivalent to the existence and uniqueness for the corresponding random PDE (5). This is because Z is the pathwise-unique strong solution [12] to the standard heat SPDE and $V+Z$ is a solution to $e_{AC}(f,u_0)$ if and only if V satisfies (4). Furthermore $Z(t,x)$ is a.s. α-Hölder-continuous with $\alpha_{\text{time}}=1/4-\varepsilon$ and $\alpha_{\text{space}}=1/2-\varepsilon$, $\forall\varepsilon>0$ [12]. For the rest of the proof, we fix $\omega\in\Omega$, and treat the path-by-path deterministic version of our random PDE (5). Following the proof in Temam [11], Chapter III in Theorem 1.1, we have $\mathbb{P}$-a.s. that there is a unique continuous (in (t,x)) solution V to (5) satisfying (4) if $u_0\in C([0,L])$. In fact, this implies that $|f(V+Z)|\leqslant M<\infty$ on $\mathcal{R}_{t,L}$; thus $I_0(t,\cdot)\in C^1(0,L)$ with $|DI_0(t,x)|\leqslant CMt^{1/2}$ for $t>0$ (I_0's smoothness is obtained throughout as in Theorems 2 to 5 in Chapter 1 of [9]) hence $V(t,\cdot)\in C^1(0,L)$ ($H(t,\cdot)$ in (4) is in $C^2(0,L)$). If additionally u_0 is Lipschitz on [0,L]; then $f(V+Z)$ is Hölder continuous on $[0,L]$, uniformly locally in t, which implies that $I_0(t,\cdot)\in C^2(0,L)$ hence $V(t,\cdot)\in C^2(0,L)$ $\forall t>0$. Also, a.s. $\mathbb{P}$:

(a) for all $0\leqslant s<T$ and all $u_0\in L^2(0,L)$, there exists a unique solution

$$
V\in C\big([s,\infty);L^2(0,L)\big)\cap L^2\big(s,T;H_0^1(0,L)\big)\cap L^{2p}\big(s,T;L^{2p}(0,L)\big),\quad \text{with } V(s)=u_0;
$$

(b) if $u_0\in H_0^1(0,L)$, then $V\in C([s,\infty);H_0^1(0,L))\cap L^2(s,T;H^2(0,L))$,

and hence, $V\in C([s+\varepsilon,\infty);H_0^1(0,L))\cap L^2(s+\varepsilon,\infty;H^2(0,L))$, for every $u_0\in L^2(0,L)$ and $\varepsilon>0$, where $H_0^1(0,L):=\{v\in H^1(0,L), v(0)=v(L)=0\}$ and $H^2(0,L)$ are the usual Sobolev spaces. It is then clear that $U(t,x)=V(t,x)+Z(t,x)$ is the pathwise-unique (because uniqueness holds a.s. for both V and Z) strong solution (because the white noise $\mathcal{W}$ is fixed throughout) of (1), and that U is $\mathbb{P}$ a.s. continuous under our conditions on u_0 (since both V and Z are).

We now sketch the steps of proving the existence of our random attractor $\mathcal{A}_{\Phi_U}$. In light of Remark 1, we look at our white noise $\mathcal{W}$ as a two-sided (in time) space–time white noise on $(\Omega,\mathcal{F},\mathbb{P})$. I.e., if $\Omega=\{\omega\in C(\mathbb{R}\times[0,L];\mathbb{R})\colon \omega(0,x)=\omega(t,0)=0\}$ with $\mathbb{P}$ being the product measure of two Brownian-sheet measures on the negative and positive time parts of Ω; then $W(t,x)=\omega(t,x)$ and $\mathcal{W}$ is the white noise corresponding to W. It can easily be checked that Φ_U satisfies properties (i) and (ii). Let $\beta>0$; then, as above, (5) has a unique solution V_β with the same regularity as V for all $-\infty<s<T$ (since Z_β is Hölder continuous). Multiplying by V_β^{2p-1} and integrating over $[0,L]$ in (5); using Young's and Hölder's inequalities repeatedly and the generalized Poincaré inequality $|v|_{L^p}\leqslant L|Dv|_{L^p}$, $p\geqslant 1$ along with its consequence $\int_0^L v^{2p-1}\Delta_x v\,\mathrm{d}x\leqslant -((2p-1)/p^2L)|v|_{L^{2p}}^{2p}$; and using elementary inequalities on f and elementary manipulations, we get (with $L^{2p}=L^{2p}(0,L)$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|V_\beta(t)\right|_{L^{2p}}^{2p}+\lambda\left|V_\beta(t)\right|_{L^{2p}}^{2p}\leqslant\frac{\beta}{\varepsilon^{2p-1}}\left|Z_\beta(t)\right|_{L^{2p}}^{2p}+2p\sum_{\substack{0\leqslant i\leqslant 2p-2\\ i\ \text{even}}}\binom{2p-1}{i}c_{0,i}\left|Z_\beta(t)\right|_{L^i}^{i}$$
$$+2p\Biggl\{\sum_{\substack{1\leqslant i\leqslant 2p-1\\ i\ \text{odd}}}\binom{2p-1}{i}\biggl[K_{0,i}L^{(4p-i-2)/(4p-2)}\left|Z_\beta(t)\right|_{L^{4p-2}}^{i}+\frac{iK_{1,i}}{(4p-2)\varepsilon_i^{(4p-i-2)/i}}\left|Z_\beta(t)\right|_{L^{4p-2}}^{4p-2}\biggr]\Biggr\}, \tag{6}$$

where $|Z_\beta(t)|_{L^0}^0 \triangleq 1$ and $\lambda=\lambda_1/2$ (λ_1 is the first positive eigenvalue for the Laplace operator). Now, picking β large enough (similarly to [7,8]), using Gronwall's lemma, and letting $P(t)$ denote the right-hand side of (6); we can deduce that there exists an $s_1(\omega)$ such that if $s<s_1(\omega)$ and $-1\leqslant t\leqslant 0$,

$$\left|V_\beta(t)\right|_{L^{2p}}^{2p}\leqslant\kappa\left[\left|V_\beta(s)\right|_{L^{2p}}^{2p}\mathrm{e}^{\lambda s}+\int_{-\infty}^{0}P(r)\,\mathrm{e}^{\lambda r}\,\mathrm{d}r\right]\leqslant r_0(\omega)=1+\kappa\int_{-\infty}^{0}P(r)\,\mathrm{e}^{\lambda r}\,\mathrm{d}r. \tag{7}$$

The at-most-polynomial growth of the norms of $Z_\beta(t)$ in $P(t)$ in (6) as $t\to-\infty$ (and hence the finiteness of $r_0=r_0(\omega)$) follows straightforwardly from standard results concerning the elementary SPDE corresponding to Z_β (e.g., see Lemma 4.1 and the ensuing discussion in [8] as well as [7,12]). On the other hand, if $\mathbb{A}=-\Delta_x$ (the negative Dirichlet Laplacian), then $\mathrm{e}^{-t\mathbb{A}}v(x)=\int_0^L v(y)G_t(x,y)\,\mathrm{d}y$ for $t>0$, and so using (4) on the interval $[-1,0]$ and applying the operator $\mathbb{A}^{1/8}$ gives us

$$\begin{aligned}\left|\mathbb{A}^{1/8}V_\beta(0)\right|_{L^2}&\leqslant\left|\mathbb{A}^{1/8}\mathrm{e}^{-\mathbb{A}}V_\beta(-1)\right|_{L^2}+\int_{-1}^{0}\left\{\left|\mathbb{A}^{1/8}\mathrm{e}^{\mathbb{A}s}f\bigl(V_\beta(s)+Z_\beta(s)\bigr)\right|_{L^2}+\beta\left|\mathbb{A}^{1/8}\mathrm{e}^{\mathbb{A}s}Z_\beta(s)\right|_{L^2}\right\}\mathrm{d}s\\ &\leqslant R_0(\omega)=\kappa_0 r_0^{1/(2p)}(\omega)+\kappa_1\int_{-1}^{0}\bigl(|s|^{-3/8}+1\bigr)\bigl[r_0^{(2p-1)/(2p)}(\omega)\\ &\qquad+\left|Z_\beta(s)\right|_{L^{2p-1}}^{2p-1}+1+\beta\left|Z_\beta(s)\right|_{L^1}\bigr]\,\mathrm{d}s,\end{aligned} \tag{8}$$

where the constants κ_0,κ_1 depend on L and p. To obtain the last inequality, we used Hölder inequality, the Sobolev embedding, and the smoothing properties of $(\mathrm{e}^{-t\mathbb{A}})_{t\geqslant 0}$ and $\mathrm{e}^{-\mathbb{A}}$:

$$\begin{aligned}&|z|_{L^2}\leqslant C_2|z|_{W^{1/2,1}}\quad\forall z\in W^{1/2,1}(0,L),\\ &\left|\mathrm{e}^{-\mathbb{A}t}z\right|_{W^{s_2,r}}\leqslant C_1\bigl(t^{(s_1-s_2)/2}+1\bigr)|z|_{W^{s_1,r}}\quad\forall z\in W^{s_1,r}(0,L),\ -\infty<s_1\leqslant s_2<\infty,\ r\geqslant 1,\\ &\left|\mathbb{A}^{1/8}\mathrm{e}^{-\mathbb{A}}\right|_{\mathcal{L}(L^2(0,L))}\leqslant C_0\end{aligned} \tag{9}$$

with $r=1,\ s_1=-1/4,\ s_2=1/2$.

Lastly, if we let $\mathcal{K}(\omega)$ be the ball in $\mathcal{D}(\mathbb{A}^{1/8})$ of radius $R_0(\omega)+|\mathbb{A}^{1/8}Z_\beta(0,\omega)|_{L^2}$; then $\mathcal{K}(\omega)$ is compact because $\mathbb{A}$ has a compact inverse, and it is obviously an attracting set at time 0. As in Crauel and Flandoli [6] (Theorem 3.11), the existence of our random attractor is concluded from the existence of the random compact absorbing set in $L^2(0,L)$. □

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Appendix

We collect here definitions and conventions that are used throughout this article. Filtrations are assumed to satisfy the usual conditions (completeness and right continuity), and any probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ with such a filtration is termed a usual probability space.

Definition A.2 (*Strong and weak solutions to* $e_{AC}(f, u_0)$). We say that the pair $(U, \mathcal{W})$ defined on the usual probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ is a continuous solution to the stochastic *PDE* $e_{AC}(f, u_0)$ if $\mathcal{W}$ is a space–time white noise on $\mathcal{C}_L$; the random field $U(t, x)$ is predictable (as in [12]), with continuous paths on $\mathcal{C}_L$; and the pair $(U, \mathcal{W})$ satisfies either the test function formulation:

$$\big(U(t) - u_0, \varphi\big) - \int_0^t \big(U(s), \varphi''\big)\,\mathrm{d}s = \int_0^t \big(f\big(U(s)\big), \varphi\big)\,\mathrm{d}s + \int_0^L\!\!\int_0^t \varphi(x)\mathcal{W}(\mathrm{d}s, \mathrm{d}x);\quad 0 \leqslant t < \infty,\ \text{a.s. } \mathbb{P}, \tag{TFF}$$

for every $\varphi \in \Theta_0^L \triangleq \{\varphi \in C_c^\infty(\mathbb{R}; \mathbb{R})\colon \varphi(0) = \varphi(L) = 0\}$, where $(\cdot, \cdot)$ is the L^2 inner product on $[0, L]$; or the Green function formulation

$$U(t, x) = \int_0^L\!\!\int_0^t G_{t-s}(x, y)\big[f\big(U(s, y)\big)\,\mathrm{d}s\,\mathrm{d}y + \mathcal{W}(\mathrm{d}s, \mathrm{d}y)\big] + \int_0^L G_t(x, y)u_0(y)\,\mathrm{d}y;\quad s < t < \infty,\ \text{a.s. } \mathbb{P}, \tag{GFF}$$

where $G_t(x, y)$ is the fundamental solution to the deterministic Dirichlet heat equation. A solution is termed strong if the white noise $\mathcal{W}$ and the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ are fixed a priori and $\mathcal{F}_t$ is the augmentation of the natural filtration for $\mathcal{W}$ under $\mathbb{P}$. It is termed a weak solution if we are allowed to choose the probability space and the white noise $\mathcal{W}$ on it, without requiring that the filtration be the augmented natural filtration of $\mathcal{W}$. Finally, we say that pathwise uniqueness holds for $e_{AC}(f, u_0)$ if whenever $(U^{(1)}, \mathcal{W})$ and $(U^{(2)}, \mathcal{W})$ are both solutions to $e_{AC}(f, u_0)$ on the same $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ then $U^{(1)}$ and $U^{(2)}$ are $\mathbb{P}$ indistinguishable; i.e., $\mathbb{P}[U^{(1)}(t, x) = U^{(2)}(t, x); t \in (0, \infty), x \in [0, L]] = 1$. We often simply say that U solves $e_{AC}(f, u_0)$ (weakly or strongly) to mean the same thing as above.

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