## Algebraic Geometry

# The elliptic $K 3$ surfaces with a maximal singular fibre 

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Received 20 June 2003; accepted 15 July 2003
Presented by Michel Raynaud


#### Abstract

We give the defining equation of complex elliptic $K 3$ surfaces with a maximal singular fibre. Then we study the reduction modulo $p$ at a particularly interesting prime p. To cite this article: T. Shioda, C. R. Acad. Sci. Paris, Ser. I 337 (2003). © 2003 Académie des sciences. Published by Éditions scientifiques et médicales Elsevier SAS. All rights reserved.


## Résumé

Des surfaces $K 3$ elliptiques possédant une fibre singulière maximale. Nous donnons l'équation des surfaces $K 3$ elliptiques possédant une fibre singulière maximale. Puis nous étudions leur réduction modulo $p$, où $p$ est un nombre premier particulièrement intéressant. Pour citer cet article : T. Shioda, C. R. Acad. Sci. Paris, Ser. I 337 (2003). © 2003 Académie des sciences. Published by Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

## Version française abrégée

Soit $f, g$ deux polynômes complexes d'une variable $t$. Considérons la courbe elliptique $E_{f, g}$ d'équation $y^{2}=x^{3}-3 f(t) x-2 g(t)$. Le discriminant est $\Delta=c \cdot h$, où $h:=f^{3}-g^{2}$ et $c$ est une constante. Soit $S_{f, g}$ la surface elliptique sur $\mathbf{P}^{1}$ définie par la même équation.

Pour une surface $K 3$ elliptique complexe, la fibre singulière maximale est de type $I_{19}$ ou $I_{14}^{*}$ (avec notation de Kodaira). L'existence d'une surface $K 3$ elliptique possédant une fibre singulière de type $I_{19}$ (ou $I_{14}^{*}$ ) est connue par Miranda et Persson [10] (ou Nishiyama [11]).

Dans cette Note, nous démontrons l'unicité et donnons l'équation explicite de telles surfaces (Théorèmes 1.1 et 1.2). En effet, la question est équivalente à la détermination de trois polynômes $\{f, g, h\}$ (tels que $h=f^{3}-g^{2}$ ) de degrés $2 m, 3 m, m+1$ respectivement pour $m=4$ ou $m=3$; nous les appelons «Davenport-Stothers triple» d'ordre $m$ [18]. La démonstration est basée sur le résultat de Stothers [19], et l'équation explicite de $f, g$ est due à Hall [5] ou Birch [3]. L'idée clef est inspirée par le théorème de Shafarevich qui considère la formule du discriminant comme l'équation de seconde courbe elliptique $Y^{2}=X^{3}+c^{\prime} \Delta$.

[^0]Puis nous étudions la réduction modulo $p$ d'une telle surface $K 3$ elliptique, pour un nombre premier $p$ particulièrement intéressant. Cette réduction est une surface $K 3$ elliptique supersingulière du rang positif. On en détermine la structure de réseau de Mordell-Weil avec les générateurs explicites (Théorèmes 3.1 et 3.2).

Finalement une conjecture est énoncée pour la réduction supersingulière d'une surface $K 3$ «singulière».

## 1. Introduction

Given complex polynomials $f, g \in \mathbf{C}[t]$ with $h:=f^{3}-g^{2} \neq 0$, let

$$
E_{f, g}: y^{2}=x^{3}-3 f(t) x-2 g(t)
$$

be an elliptic curve over $\mathbf{C}(t)$. The discriminant $\Delta=\Delta\left(E_{f, g}\right)$ is given by

$$
\Delta=4(-3 f)^{3}+27(2 g)^{2}=-4 \cdot 3^{3}\left(f^{3}-g^{2}\right)=-108 \cdot h
$$

and the absolute invariant $j\left(E_{f, g}\right)$ is equal to the rational function $J$ :

$$
J=\frac{f^{3}}{h}, \quad \text { with } J-1=\frac{g^{2}}{h}
$$

Let $S_{f, g}$ be the elliptic surface (with a section) over $\mathbf{P}^{1}$ associated with $E_{f, g}$.
Now for an elliptic surface with a section (over $\mathbf{P}^{1}$ ), we have the well known Picard number formula (cf. [15,16])

$$
\rho=r+2+\sum_{v}\left(m_{v}-1\right)
$$

where $r$ is the Mordell-Weil rank and $m_{v}$ is the number of irreducible components of the fibre at $v$, the summation running over all $v \in \mathbf{P}^{1}$. For an elliptic $K 3$ surface over $\mathbf{C}$ (complex numbers), the Hodge bound $\rho \leqslant h^{1,1}=20$ gives the upper bound of $m_{v}$, i.e., $m_{v} \leqslant 19$. The maximal case $m_{v}=19$ can occur only for a singular fibre of type $I_{19}$ or of type $I_{14}^{*}$ in Kodaira's notation [9], and when this occurs, all the other singular fibres must be irreducible ( $m_{v^{\prime}}=1$ for $v^{\prime} \neq v$ ) and we have $\rho=20, r=0$.

Theorem 1.1. Suppose $S$ is an elliptic $K 3$ surface with a section having a singular fibre of type $I_{19}$. Then $S$ is unique up to isomorphism, having five other singular fibres of type $I_{1}$, and it is isomorphic to $S_{f, g}$ where $\{f, g, h\}$ is a triple given by Hall [5, p. 185]:

$$
\begin{aligned}
f= & t^{8}+6 t^{7}+21 t^{6}+50 t^{5}+86 t^{4}+114 t^{3}+109 t^{2}+74 t+28 \\
g= & 1 / 2 \cdot\left(2 t^{12}+18 t^{11}+90 t^{10}+312 t^{9}+816 t^{8}+1692 t^{7}+2832 t^{6}\right. \\
& \left.+3864 t^{5}+4272 t^{4}+3746 t^{3}+2517 t^{2}+1167 t+299\right) \\
h= & -27 / 4 \cdot\left(4 t^{5}+15 t^{4}+38 t^{3}+61 t^{2}+62 t+59\right)
\end{aligned}
$$

Theorem 1.2. Suppose $S$ is an elliptic $K 3$ surface with a section having a singular fibre of type $I_{14}^{*}$. Then $S$ is unique up to isomorphism, having four other singular fibres of type $I_{1}$, and it is isomorphic to $S_{f, g}$ where $\{f, g, h\}$ is a triple given by Birch [3, p. 65]:

$$
f=t^{6}+4 t^{4}+10 t^{2}+6, \quad g=t^{9}+6 t^{7}+21 t^{5}+35 t^{3}+63 / 2 t, \quad h=27 t^{4}+351 / 4 t^{2}+216
$$

We note that the existence of an elliptic $K 3$ surface with five $I_{1}$ and $I_{19}$ is shown by Miranda-Persson via transcendental method (see the first of their list in [10]). The existence of one with $I_{14}^{*}$ is shown by Nishiyama [11] via lattice-theoretic method. The above theorems give an explicit defining equation of such a surface, together with uniqueness.

If there is anything new in our approach, it is the idea to relate the two independently studied subjects "elliptic surfaces" and "integral points" by a link inspired by Shafarevich's famous theorem. Namely its proof regards the formula of the discriminant as defining a second elliptic curve $Y^{2}=X^{3}+c^{\prime} \Delta$, of which the pair $(f, g)$ is an "integral point". This simple idea is surprisingly useful (cf. [17], [18] in preparation), and the above results are just some of first examples.

## 2. Proof

We freely use Kodaira's general theory of elliptic surfaces [9].
To prove Theorem 1.1, we may assume that the singular fibre of type $I_{19}$ lies over $t=\infty$. Then we write the generic fibre $E$ of $S$ in the Weierstrass form $E=E_{f, g}$ for some $f, g$, where we have $\operatorname{deg}(f) \leqslant 8, \operatorname{deg}(g) \leqslant 12$ since $S$ is a $K 3$ surface. We have $J=f^{3} / h$ and $J$ has a pole of order 19 at $t=\infty$. Hence $\operatorname{deg}(f)=8$ and $\operatorname{deg}(h)=24-19=5$, since a $K 3$ has the Euler number 24 . Thus $f, g, h$ have respectively degree $2 m, 3 m, m+1$ for $m=4$, i.e., $\{f, g, h\}$ is a Davenport-Stothers triple of order $m=4$ in the sense of [18]. By Stothers [19, p. 364], such a triple is essentially unique for $m=1,2,3,4$. Since Hall's data gives such a triple for $m=4$, Theorem 1.1 is proven. (All the other singular fibres are of type $I_{1}$ since $h$ has only simple zeros.)

Similarly, to prove Theorem 1.2, we assume that the singular fibre of type $I_{14}^{*}$ lies over $t=\infty$. Then we can write the generic fibre $E$ of $S$ in the Weierstrass form $E=E_{f, g}$ for some $f, g$, where we have $\operatorname{deg}(f)=6, \operatorname{deg}(g)=9$ and $\operatorname{deg}(h)=24-20=4$, since $I_{14}^{*}$ has the local Euler number 20. Thus $f, g, h$ has respectively degree $2 m, 3 m, m+1$ for $m=3$, i.e., $\{f, g, h\}$ is a Davenport-Stothers triple of order $m=3$. By Stothers [19, p. 364], it is essentially unique. Since Birch's data gives such a triple, this completes the proof of Theorem 1.2.

## 3. Further properties

Let us denote the (unique) elliptic $K 3$ surface of Theorem 1.1 or Theorem 1.2 by

$$
X=S_{19} \quad \text { or } \quad S_{14}^{*}
$$

Then both surfaces are "singular" $K 3$ surfaces in the sense that $\rho=h^{1,1}=20$. By [7], such a surface $X$ determines and is determined by a positive-definite two by two even matrix, say $Q_{X}$, the intersection matrix on the lattice $T_{X}$ of transcendental cycles. In the case under consideration, we have

$$
Q_{X}=\left(\begin{array}{cc}
2 & 1 \\
1 & 10
\end{array}\right) \quad \text { or } \quad\left(\begin{array}{ll}
2 & 0 \\
0 & 2
\end{array}\right)
$$

The Néron-Severi lattice $N=N S(X)$ is given by

$$
N=U \oplus A_{18}^{-} \quad \text { or } \quad U \oplus D_{18}^{-},
$$

where $U$ is a rank 2 hyperbolic lattice and $A_{n}^{-}$(or $D_{n}^{-}$) stands for the (negative-definite) root lattice of type $A_{n}$ or $D_{n}$.

As shown in [7], any singular $K 3$ surface can be defined over an algebraic number field, and its reduction modulo a prime is, if not a bad reduction, either a singular $(\rho=20)$ or supersingular $(\rho=22) K 3$ surface, and both cases occur with positive density. In our case, both $X=S_{19}, S_{14}^{*}$ are defined over the rational number field $\mathbf{Q}$. For any prime number $p>3$, the reduction of $X$ modulo $p, X(p)$, is an elliptic $K 3$ surface over $\mathbf{P}^{1}$, for which the types of the singular fibres are the same as in the complex case except for the following cases:

$$
X=S_{19}, p=19 \quad \text { or } \quad X=S_{14}^{*}, p=7
$$

Some new interesting feature appears in these exceptional cases:
Theorem 3.1. Let $X=S_{19}$, and consider its reduction $X(p)$ modulo $p=19$. Then (i) it is an elliptic $K 3$ surface defined over the finite field $\mathbf{F}_{p}=\mathbf{Z} / p \mathbf{Z}$ with defining equation:

$$
y^{2}=x^{3}-3(t+3)(t-5)^{7} x-2(t+3)^{11}(t-5)
$$

We have the discriminant $\Delta=(t+3)^{3}(t-5)^{2}$ and $J=(t-5)^{19}$ up to constants. There are three singular fibres of type $I_{19}$, II and III at $t=\infty, 5,-3$ respectively and the trivial lattice is $V=U \oplus A_{18}^{-} \oplus A_{1}^{-}$, with rk $V=21$. (ii) The Néron-Severi lattice $N=N S(X(p))$ has rank $\rho=22$ (i.e., supersingular) and $|\operatorname{det} N|=p^{2}$. (iii) The Mordell-Weil lattice is a lattice of rank one generated by a rational point $P$ of height $\langle P, P\rangle=19 / 2$. Explicitly, $P$ is given by

$$
P=\left(\frac{12(t+3)^{10}}{(t-5)^{6}}, \frac{\mathrm{i}(t+3)^{15}}{(t-5)^{9}}\right)
$$

Proof. It is straightforward to verify (i) by reducing the data in Theorem 1.1 modulo $p=19$ (cf. [20]). For proving (ii) and (iii), we first observe that the $J$-function defines a purely inseparable map of degree $p$, and that $X(p)$ is obtained as the base change via $T=t^{p}$ of a rational elliptic surface with three singular fibres $I_{1}, I I$ and $I I I^{*}$. Indeed, multiply $(t+3)^{84}(t-5)^{18}$ to the both side of the defining equation, and rewrite the resulting equation in terms of

$$
X=x(t+3)^{28}(t-5)^{6}, \quad Y=y(t+3)^{42}(t-5)^{9}, \quad T=t^{p}
$$

Then we have

$$
Y^{2}=X^{3}-3(T+3)^{3}(T-5) X-2(T+3)^{5}(T-5)
$$

which defines a rational elliptic surface over the $T$-line, say $Z$, with the discriminant $\Delta=(T+3)^{9}(T-5)^{2}$ and $J=(T-5)$ up to constants. Hence $Z$ has the singular fibres $I_{1}, I I$ and $I I I^{*}$ at $T=\infty, 5,-3$ respectively. By [12] (Case No. 43), the Mordell-Weil lattice is isomorphic to $A_{1}^{*}$, i.e., a rank one lattice generated by a minimal vector $Q$ of height $1 / 2$. It is easy to determine such, and we have

$$
Q=\left(12(T+3)^{2}, \mathrm{i}(T+3)^{3}\right) \quad\left(\mathrm{i}^{2}=-1\right)
$$

The rational point $P$ is obtained from $Q$ via the base change $T=t^{p}$ (of degree $p$ ) and the coordinate change. The height of $P$ is equal to $\langle P, P\rangle=p \cdot\langle Q, Q\rangle=p / 2$ (see [16, Proposition 8.12]).

Now we claim that $N=N S(X(p))$ is generated by the divisor of section $(P)$ and the trivial sublattice $V$. In fact, let $N_{1}$ denote the sublattice of $N$ generated by $(P)$ and $V$, and let $N_{2}$ be the sublattice of $N_{1}$ generated by $(2 P)$ and $V$. Then $N_{2}$ has rank 22 and $|\operatorname{det}|=(2 \cdot 19)^{2}$, because $|\operatorname{det} V|=2 \cdot 19$ and $2 P$ has height $2^{2} \cdot\langle P, P\rangle=2 \cdot 19$. Since the index $\left[N_{1}: N_{2}\right]=2$, we have $\left|\operatorname{det} N_{1}\right|=\left|\operatorname{det} N_{2}\right| / 2^{2}=19^{2}$. Letting $\nu=\left[N: N_{1}\right]$, we have $|\operatorname{det} N|=\left|\operatorname{det} N_{1}\right| / \nu^{2}=(19 / \nu)^{2}$, hence $v=1$ or 19 . If $v=19, N$ would be unimodular, and it is an even lattice with signature $(1,21)$. This is a contradiction, as $1-21=-20$ is not a multiple of 8 . Therefore we must have $v=1$, i.e., $N=N_{1}$.

This proves $|\operatorname{det} N|=19^{2}$ as stated in (ii). Also the Mordell-Weil group is isomorphic to the quotient group $N / V$ in general (see [16, Theorem 1.3]), which is generated in our case by the divisor class of $(P)$. It follows that $P$ is the generator of the Mordell-Weil group in question, proving (iii).

Theorem 3.2. Let $X=S_{14}^{*}$, and consider its reduction $X(p)$ modulo $p=7$. Then (i) it is an elliptic $K 3$ surface defined over the finite field $\mathbf{F}_{p}=\mathbf{Z} / p \mathbf{Z}$ with defining equation:

$$
y^{2}=x^{3}-3(t+1)^{3}(t-1)^{3} x-2 t^{7}(t+1)(t-1)
$$

We have $\Delta=(t+1)^{2}(t-1)^{2}$ and $J=(t+1)^{7}(t-1)^{7}$ up to constants. There are three singular fibres of type $I_{14}^{*}$, II, at $t=\infty, 1,-1$ respectively; the trivial lattice is $V=U \oplus D_{18}^{-}$, with $\mathrm{rk} V=20$. (ii) The Néron-Severi lattice $N=N S(X(p))$ has rank $\rho=22$ (i.e., supersingular) and $|\operatorname{det} N|=p^{2}$. (iii) The Mordell-Weil lattice is a lattice of rank two generated by two rational points $P_{1}, P_{2}$ of height $7 / 2$ such that $\left\langle P_{1}, P_{2}\right\rangle=0$. Explicitly, they are given by

$$
P_{1}=\left(\frac{3-t^{7}}{\left(t^{2}-1\right)^{2}}, \frac{1}{\left(t^{2}-1\right)^{3}}\right), \quad P_{2}=\left(\frac{4-t^{7}}{\left(t^{2}-1\right)^{2}}, \frac{\mathrm{i}}{\left(t^{2}-1\right)^{3}}\right)
$$

We omit the proof which is similar to the above proof of Theorem 3.1.

## 4. Remarks

(1) As mentioned before, we call ([18]) a polynomial triple $\{f, g, h\}$ a Davenport-Stothers triple of degree $m$ if $f^{3}-g^{2}=h$ and $\operatorname{deg} f=2 m$, $\operatorname{deg} g=3 m$, $\operatorname{deg} h=m+1$. In characteristic zero, it is known that $f, g, h$ are relatively prime and they have only simple zeros (see [4,19]). Then the elliptic surface $S_{f, g}$ has $m+1$ singular fibres of type $I_{1}$ and a singular fibre of type $I_{5 m-1}$ for $m$ even (resp. of type $I_{5 m-1}^{*}$ for $m$ odd). For $m=1$ (resp. $m=2$ ), the rational elliptic surface with singular fibres $I_{1}, I_{1}, I_{4}^{*}\left(\right.$ resp. $\left.I_{1}, I_{1}, I_{1}, I_{9}\right)$ has been studied by [13] (resp. [2]). For $m=3$, 4, we have the $K 3$ surfaces stated in Theorems 1.1 or 1.2. Further study on this series of elliptic surfaces for any $m$ is in preparation [18]. (We reported some of our results including Theorems 1.1 and 3.1 above at the international workshop on "Discrete Groups and Moduli" held at Nagoya University, September 2002.)
(2) For a Davenport-Stothers triple $\{f, g, h\}$ in positive characteristic, $f, g, h$ can have common factors and multiple zeros. The cases of Theorems 3.1 or 3.2 give some example to this. For this topic, compare [6].
(3) In characteristic $p>0$, we have the weaker bound $\rho \leqslant b_{2}=22$ than the Hodge bound $\rho \leqslant h^{1,1}=20$. With the notation in Introduction, we then have $m_{v} \leqslant 21$. To supplement the title of this paper, we note that equality $m_{v}=21$ (i.e., type $I_{21}$ or $I_{16}^{*}$ ) cannot occur. For instance, if there is a fibre of type $I_{21}$, then the Néron-Severi lattice $N$ contains the sublattice of finite index $U \oplus A_{20}^{-}$, hence $\operatorname{det} N$ must divide 21 . But this is impossible since $\operatorname{det} N$ is an even power of $p$ by Artin [1]. Similarly the case of $I_{16}^{*}$ is easily ruled out for $p>2$. It is also impossible for $p=2$ (see [8,14]). The case $m_{v}=20$ and $r=0$ is also easily seen to be impossible.
(4) We take this opportunity to formulate a conjecture on the relation of transcendetal cycles and supersingular reduction of a $K 3$ surface.

Let $X$ be a singular $K 3$ surface defined over an algebraic number field $K \subset \mathbf{C}$. Then the Néron-Severi lattice $N=N S(X)$ is a sublattice of the second cohomology group $H=H^{2}(X, \mathbf{Z})$ with cup product. Let $T_{X}$ be the orthogonal complement of $N$ in $H$; it is the lattice of transcendetal cycles on $X$ which is a positive-definite even lattice of rank two. On the other hand, let $X(\mathbf{p})$ be a supersingular reduction of $X$ modulo a prime ideal $\mathbf{p}$ of $K$; the reduction defines a natural embedding of $N=N S(X)$ into $N(\mathbf{p})=N S(X(\mathbf{p}))$. Let $L(\mathbf{p})$ be the orthogonal complement of $N$ in $N(\mathbf{p})$, which is a negative-definite even lattice of rank two.

Conjecture 4.1. The lattices $T_{X}$ and $L(\mathbf{p})$ are similar. In other words, $L(\mathbf{p})$ is isomorphic to $T_{X}^{-}$up to scaling.
When $X$ is an elliptic $K 3$ surface, we can formulate a similar conjecture in terms of Mordell-Weil lattices. The above Theorem 3.2(iii) can be seen as an example of this.

## Acknowledgements

I would like to thank Michel Raynaud for his interest in my work.

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    doi:10.1016/j.crma.2003.07.007

