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C. R. Acad. Sci. Paris, Ser. I 337 (2003) 461-466

Algebraic Geometry

The elliptic K3 surfaces with a maximal singular fibre

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Received 20 June 2003; accepted 15 July 2003
Presented by Michel Raynaud

Abstract

We give the defining equation of complex elliptic *K*3 surfaces with a maximal singular fibre. Then we study the reduction modulo *p* at a particularly interesting prime *p*. *To cite this article: T. Shioda, C. R. Acad. Sci. Paris, Ser. I* 337 (2003). © 2003 Académie des sciences. Published by Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

Résumé

Des surfaces K3 elliptiques possédant une fibre singulière maximale. Nous donnons l'équation des surfaces K3 elliptiques possédant une fibre singulière maximale. Puis nous étudions leur réduction modulo p, où p est un nombre premier particulièrement intéressant. *Pour citer cet article : T. Shioda, C. R. Acad. Sci. Paris, Ser. I 337 (2003).*© 2003 Académie des sciences. Published by Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

Version française abrégée

Soit f, g deux polynômes complexes d'une variable t. Considérons la courbe elliptique $E_{f,g}$ d'équation $y^2 = x^3 - 3f(t)x - 2g(t)$. Le discriminant est $\Delta = c \cdot h$, où $h := f^3 - g^2$ et c est une constante. Soit $S_{f,g}$ la surface elliptique sur \mathbf{P}^1 définie par la même équation.

Pour une surface K3 elliptique complexe, la fibre singulière maximale est de type I_{19} ou I_{14}^* (avec notation de Kodaira). L'existence d'une surface K3 elliptique possédant une fibre singulière de type I_{19} (ou I_{14}^*) est connue par Miranda et Persson [10] (ou Nishiyama [11]).

Dans cette Note, nous démontrons l'unicité et donnons l'équation explicite de telles surfaces (Théorèmes 1.1 et 1.2). En effet, la question est équivalente à la détermination de trois polynômes $\{f,g,h\}$ (tels que $h=f^3-g^2$) de degrés 2m, 3m, m+1 respectivement pour m=4 ou m=3; nous les appelons «Davenport-Stothers triple» d'ordre m [18]. La démonstration est basée sur le résultat de Stothers [19], et l'équation explicite de f, g est due à Hall [5] ou Birch [3]. L'idée clef est inspirée par le théorème de Shafarevich qui considère la formule du discriminant comme l'équation de seconde courbe elliptique $Y^2 = X^3 + c'\Delta$.

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Puis nous étudions la réduction modulo p d'une telle surface K3 elliptique, pour un nombre premier p particulièrement intéressant. Cette réduction est une surface K3 elliptique supersingulière du rang positif. On en détermine la structure de réseau de Mordell-Weil avec les générateurs explicites (Théorèmes 3.1 et 3.2).

Finalement une conjecture est énoncée pour la réduction supersingulière d'une surface K3 « singulière ».

1. Introduction

Given complex polynomials $f, g \in \mathbb{C}[t]$ with $h := f^3 - g^2 \neq 0$, let

$$E_{f,g}: y^2 = x^3 - 3f(t)x - 2g(t)$$

be an elliptic curve over C(t). The discriminant $\Delta = \Delta(E_{f,g})$ is given by

$$\Delta = 4(-3f)^3 + 27(2g)^2 = -4 \cdot 3^3(f^3 - g^2) = -108 \cdot h,$$

and the absolute invariant $j(E_{f,g})$ is equal to the rational function J:

$$J = \frac{f^3}{h}$$
, with $J - 1 = \frac{g^2}{h}$.

Let $S_{f,g}$ be the elliptic surface (with a section) over \mathbf{P}^1 associated with $E_{f,g}$. Now for an elliptic surface with a section (over \mathbf{P}^1), we have the well known Picard number formula (cf. [15,16])

$$\rho = r + 2 + \sum_{v} (m_v - 1),$$

where r is the Mordell-Weil rank and m_v is the number of irreducible components of the fibre at v, the summation running over all $v \in \mathbf{P}^1$. For an elliptic K3 surface over C (complex numbers), the Hodge bound $\rho \leqslant h^{1,1} = 20$ gives the upper bound of m_v , i.e., $m_v \le 19$. The maximal case $m_v = 19$ can occur only for a singular fibre of type I_{19} or of type I_{14}^* in Kodaira's notation [9], and when this occurs, all the other singular fibres must be irreducible $(m_{v'}=1 \text{ for } v'\neq v)$ and we have $\rho=20,\ r=0.$

Theorem 1.1. Suppose S is an elliptic K3 surface with a section having a singular fibre of type I_{19} . Then S is unique up to isomorphism, having five other singular fibres of type I_1 , and it is isomorphic to $S_{f,g}$ where $\{f,g,h\}$ is a triple given by Hall [5, p. 185]:

$$f = t^8 + 6t^7 + 21t^6 + 50t^5 + 86t^4 + 114t^3 + 109t^2 + 74t + 28,$$

$$g = 1/2 \cdot (2t^{12} + 18t^{11} + 90t^{10} + 312t^9 + 816t^8 + 1692t^7 + 2832t^6 + 3864t^5 + 4272t^4 + 3746t^3 + 2517t^2 + 1167t + 299),$$

$$h = -27/4 \cdot (4t^5 + 15t^4 + 38t^3 + 61t^2 + 62t + 59).$$

Theorem 1.2. Suppose S is an elliptic K3 surface with a section having a singular fibre of type I_{14}^* . Then S is unique up to isomorphism, having four other singular fibres of type I_1 , and it is isomorphic to $S_{f,g}$ where $\{f,g,h\}$ is a triple given by Birch [3, p. 65]:

$$f = t^6 + 4t^4 + 10t^2 + 6$$
, $g = t^9 + 6t^7 + 21t^5 + 35t^3 + 63/2t$, $h = 27t^4 + 351/4t^2 + 216$.

We note that the existence of an elliptic K3 surface with five I_1 and I_{19} is shown by Miranda-Persson via transcendental method (see the first of their list in [10]). The existence of one with I_{14}^* is shown by Nishiyama [11] via lattice-theoretic method. The above theorems give an explicit defining equation of such a surface, together with uniqueness.

If there is anything new in our approach, it is the idea to relate the two independently studied subjects "elliptic surfaces" and "integral points" by a link inspired by Shafarevich's famous theorem. Namely its proof regards the formula of the discriminant as defining a second elliptic curve $Y^2 = X^3 + c'\Delta$, of which the pair (f, g) is an "integral point". This simple idea is surprisingly useful (cf. [17], [18] in preparation), and the above results are just some of first examples.

2. Proof

We freely use Kodaira's general theory of elliptic surfaces [9].

To prove Theorem 1.1, we may assume that the singular fibre of type I_{19} lies over $t = \infty$. Then we write the generic fibre E of S in the Weierstrass form $E = E_{f,g}$ for some f, g, where we have $\deg(f) \le 8$, $\deg(g) \le 12$ since S is a K3 surface. We have $J = f^3/h$ and J has a pole of order 19 at $t = \infty$. Hence $\deg(f) = 8$ and $\deg(h) = 24 - 19 = 5$, since a K3 has the Euler number 24. Thus f, g, h have respectively degree 2m, 3m, m+1 for m=4, i.e., $\{f,g,h\}$ is a Davenport–Stothers triple of order m=4 in the sense of [18]. By Stothers [19, p. 364], such a triple is essentially unique for m=1,2,3,4. Since Hall's data gives such a triple for m=4, Theorem 1.1 is proven. (All the other singular fibres are of type I_1 since h has only simple zeros.)

Similarly, to prove Theorem 1.2, we assume that the singular fibre of type I_{14}^* lies over $t = \infty$. Then we can write the generic fibre E of S in the Weierstrass form $E = E_{f,g}$ for some f, g, where we have $\deg(f) = 6$, $\deg(g) = 9$ and $\deg(h) = 24 - 20 = 4$, since I_{14}^* has the local Euler number 20. Thus f, g, h has respectively degree 2m, 3m, m+1 for m=3, i.e., $\{f, g, h\}$ is a Davenport–Stothers triple of order m=3. By Stothers [19, p. 364], it is essentially unique. Since Birch's data gives such a triple, this completes the proof of Theorem 1.2.

3. Further properties

Let us denote the (unique) elliptic K3 surface of Theorem 1.1 or Theorem 1.2 by

$$X = S_{19}$$
 or S_{14}^* .

Then both surfaces are "singular" K3 surfaces in the sense that $\rho = h^{1,1} = 20$. By [7], such a surface X determines and is determined by a positive-definite two by two even matrix, say Q_X , the intersection matrix on the lattice T_X of transcendental cycles. In the case under consideration, we have

$$Q_X = \begin{pmatrix} 2 & 1 \\ 1 & 10 \end{pmatrix}$$
 or $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

The Néron–Severi lattice N = NS(X) is given by

$$N = U \oplus A_{18}^-$$
 or $U \oplus D_{18}^-$,

where U is a rank 2 hyperbolic lattice and A_n^- (or D_n^-) stands for the (negative-definite) root lattice of type A_n or D_n .

As shown in [7], any singular K3 surface can be defined over an algebraic number field, and its reduction modulo a prime is, if not a bad reduction, either a singular ($\rho = 20$) or supersingular ($\rho = 22$) K3 surface, and both cases occur with positive density. In our case, both $X = S_{19}$, S_{14}^* are defined over the rational number field \mathbf{Q} . For any prime number p > 3, the reduction of X modulo p, X(p), is an elliptic K3 surface over \mathbf{P}^1 , for which the types of the singular fibres are the same as in the complex case except for the following cases:

$$X = S_{19}, p = 19$$
 or $X = S_{14}^*, p = 7$.

Some new interesting feature appears in these exceptional cases:

Theorem 3.1. Let $X = S_{19}$, and consider its reduction X(p) modulo p = 19. Then (i) it is an elliptic K3 surface defined over the finite field $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$ with defining equation:

$$y^2 = x^3 - 3(t+3)(t-5)^7x - 2(t+3)^{11}(t-5).$$

We have the discriminant $\Delta = (t+3)^3(t-5)^2$ and $J = (t-5)^{19}$ up to constants. There are three singular fibres of type I_{19} , II and III at $t = \infty, 5, -3$ respectively and the trivial lattice is $V = U \oplus A_{18}^- \oplus A_1^-$, with $\operatorname{rk} V = 21$. (ii) The Néron–Severi lattice N = NS(X(p)) has rank $\rho = 22$ (i.e., supersingular) and $|\det N| = p^2$. (iii) The Mordell–Weil lattice is a lattice of rank one generated by a rational point P of height $\langle P, P \rangle = 19/2$. Explicitly, P is given by

$$P = \left(\frac{12(t+3)^{10}}{(t-5)^6}, \frac{\mathbf{i}(t+3)^{15}}{(t-5)^9}\right).$$

Proof. It is straightforward to verify (i) by reducing the data in Theorem 1.1 modulo p = 19 (cf. [20]). For proving (ii) and (iii), we first observe that the J-function defines a purely inseparable map of degree p, and that X(p) is obtained as the base change via $T = t^p$ of a rational elliptic surface with three singular fibres I_1 , II and III^* . Indeed, multiply $(t+3)^{84}(t-5)^{18}$ to the both side of the defining equation, and rewrite the resulting equation in terms of

$$X = x(t+3)^{28}(t-5)^6$$
, $Y = y(t+3)^{42}(t-5)^9$, $T = t^p$.

Then we have

$$Y^{2} = X^{3} - 3(T+3)^{3}(T-5)X - 2(T+3)^{5}(T-5),$$

which defines a rational elliptic surface over the T-line, say Z, with the discriminant $\Delta = (T+3)^9(T-5)^2$ and J = (T-5) up to constants. Hence Z has the singular fibres I_1 , II and III^* at $T = \infty$, 5, -3 respectively. By [12] (Case No. 43), the Mordell–Weil lattice is isomorphic to A_1^* , i.e., a rank one lattice generated by a minimal vector O of height 1/2. It is easy to determine such, and we have

$$Q = (12(T+3)^2, i(T+3)^3)$$
 $(i^2 = -1).$

The rational point P is obtained from Q via the base change $T = t^p$ (of degree p) and the coordinate change. The height of P is equal to $\langle P, P \rangle = p \cdot \langle Q, Q \rangle = p/2$ (see [16, Proposition 8.12]).

Now we claim that N = NS(X(p)) is generated by the divisor of section (P) and the trivial sublattice V. In fact, let N_1 denote the sublattice of N generated by (P) and V, and let N_2 be the sublattice of N_1 generated by (2P) and V. Then N_2 has rank 22 and $|\det| = (2 \cdot 19)^2$, because $|\det V| = 2 \cdot 19$ and 2P has height $2^2 \cdot \langle P, P \rangle = 2 \cdot 19$. Since the index $[N_1 : N_2] = 2$, we have $|\det N_1| = |\det N_2|/2^2 = 19^2$. Letting $v = [N : N_1]$, we have $|\det N| = |\det N_1|/v^2 = (19/v)^2$, hence v = 1 or 19. If v = 19, N would be unimodular, and it is an even lattice with signature (1, 21). This is a contradiction, as 1 - 21 = -20 is not a multiple of 8. Therefore we must have v = 1, i.e., $N = N_1$.

This proves $|\det N| = 19^2$ as stated in (ii). Also the Mordell–Weil group is isomorphic to the quotient group N/V in general (see [16, Theorem 1.3]), which is generated in our case by the divisor class of (P). It follows that P is the generator of the Mordell–Weil group in question, proving (iii). \square

Theorem 3.2. Let $X = S_{14}^*$, and consider its reduction X(p) modulo p = 7. Then (i) it is an elliptic K3 surface defined over the finite field $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$ with defining equation:

$$y^2 = x^3 - 3(t+1)^3(t-1)^3x - 2t^7(t+1)(t-1).$$

We have $\Delta=(t+1)^2(t-1)^2$ and $J=(t+1)^7(t-1)^7$ up to constants. There are three singular fibres of type I_{14}^* , II, at $t=\infty,1,-1$ respectively; the trivial lattice is $V=U\oplus D_{18}^-$, with $\operatorname{rk} V=20$. (ii) The Néron–Severi lattice N=NS(X(p)) has rank $\rho=22$ (i.e., supersingular) and $|\det N|=p^2$. (iii) The Mordell–Weil lattice is a lattice of rank two generated by two rational points P_1 , P_2 of height 7/2 such that $\langle P_1, P_2 \rangle = 0$. Explicitly, they are given by

$$P_1 = \left(\frac{3 - t^7}{(t^2 - 1)^2}, \frac{1}{(t^2 - 1)^3}\right), \qquad P_2 = \left(\frac{4 - t^7}{(t^2 - 1)^2}, \frac{i}{(t^2 - 1)^3}\right).$$

We omit the proof which is similar to the above proof of Theorem 3.1.

4. Remarks

- (1) As mentioned before, we call ([18]) a polynomial triple $\{f, g, h\}$ a Davenport–Stothers triple of degree m if $f^3 g^2 = h$ and deg f = 2m, deg g = 3m, deg h = m + 1. In characteristic zero, it is known that f, g, h are relatively prime and they have only simple zeros (see [4,19]). Then the elliptic surface $S_{f,g}$ has m + 1 singular fibres of type I_1 and a singular fibre of type I_{5m-1} for m even (resp. of type I_{5m-1}^* for m odd). For m = 1 (resp. m = 2), the rational elliptic surface with singular fibres I_1 , I_1 , I_4 (resp. I_1 , I_1 , I_1 , I_2) has been studied by [13] (resp. [2]). For m = 3, 4, we have the K3 surfaces stated in Theorems 1.1 or 1.2. Further study on this series of elliptic surfaces for any m is in preparation [18]. (We reported some of our results including Theorems 1.1 and 3.1 above at the international workshop on "Discrete Groups and Moduli" held at Nagoya University, September 2002.)
- (2) For a Davenport–Stothers triple $\{f, g, h\}$ in positive characteristic, f, g, h can have common factors and multiple zeros. The cases of Theorems 3.1 or 3.2 give some example to this. For this topic, compare [6].
- (3) In characteristic p > 0, we have the weaker bound $\rho \le b_2 = 22$ than the Hodge bound $\rho \le h^{1,1} = 20$. With the notation in Introduction, we then have $m_v \le 21$. To supplement the title of this paper, we note that *equality* $m_v = 21$ (i.e., type I_{21} or I_{16}^*) cannot occur. For instance, if there is a fibre of type I_{21} , then the Néron–Severi lattice N contains the sublattice of finite index $U \oplus A_{20}^-$, hence det N must divide 21. But this is impossible since det N is an even power of p by Artin [1]. Similarly the case of I_{16}^* is easily ruled out for p > 2. It is also impossible for p = 2 (see [8,14]). The case $m_v = 20$ and r = 0 is also easily seen to be impossible.
- (4) We take this opportunity to formulate a conjecture on the relation of transcendetal cycles and supersingular reduction of a K3 surface.

Let X be a singular K3 surface defined over an algebraic number field $K \subset \mathbb{C}$. Then the Néron-Severi lattice N = NS(X) is a sublattice of the second cohomology group $H = H^2(X, \mathbb{Z})$ with cup product. Let T_X be the orthogonal complement of N in H; it is the lattice of transcendetal cycles on X which is a positive-definite even lattice of rank two. On the other hand, let $X(\mathbf{p})$ be a supersingular reduction of X modulo a prime ideal \mathbf{p} of K; the reduction defines a natural embedding of N = NS(X) into $N(\mathbf{p}) = NS(X(\mathbf{p}))$. Let $L(\mathbf{p})$ be the orthogonal complement of N in $N(\mathbf{p})$, which is a negative-definite even lattice of rank two.

Conjecture 4.1. The lattices T_X and $L(\mathbf{p})$ are similar. In other words, $L(\mathbf{p})$ is isomorphic to T_X^- up to scaling.

When *X* is an elliptic *K*3 surface, we can formulate a similar conjecture in terms of Mordell–Weil lattices. The above Theorem 3.2(iii) can be seen as an example of this.

Acknowledgements

I would like to thank Michel Raynaud for his interest in my work.

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