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On the meaning of Parisi's functional order parameter

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Abstract

The author has recently proved that a famous formula discovered by G. Parisi gives at any temperature the correct value for the limiting free energy of a large class of mean field models for spin glasses (a class which contains in particular the Sherrington–Kirkpatrick model). Here we prove rigorously that (generically) the "functional order parameter" occuring in this formula can be interpreted as predicted by Parisi, namely as representing the limiting distribution of the overlap of two independent configurations. ***To cite this article: M. Talagrand, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***
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Résumé

La signification du paramètre fonctionnel d'ordre de Parisi. L'auteur a récemment démontré qu'une célèbre formule de G. Parisi donne effectivement à toute température la valeur correcte de l'énergie libre limite d'une large classe de modèles de verre de spin à champ moyen, classe contenant en particulier le modèle de Sherrington–Kirkpatrick. Cette formule fait intervenir un « paramètre d'ordre fonctionnel » dont on démontre ici que (génériquement) la signification est celle prévue par la théorie de Parisi, à savoir qu'il représente la distribution limite du recouvrement de deux configurations indépendentes. ***Pour citer cet article : M. Talagrand, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***
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1. Introduction

We consider independent standard normal r.v. $g_{i_1,\dots,i_p}$ for each integers $p,\ i_1,\dots,i_p$. For $\boldsymbol{\sigma} \in \Sigma_N = \{-1,1\}^N$, we consider the Hamiltonian of the p-spin interaction model,

$$-H_{N,p}(\boldsymbol{\sigma}) = \sqrt{\frac{p!}{N^{p-1}}} \sum_{1\leqslant i_1<\dots<i_p\leqslant N} g_{i_1,\dots,i_p}\sigma_{i_1}\cdots\sigma_{i_p}. \tag{1}$$

Consider a sequence $\boldsymbol{\beta} = (\beta_p)_{p\geqslant 1}$ with $\|\boldsymbol{\beta}\|_2^2 = \sum_{p\geqslant 1}\beta_p^2 < \infty$ and the Hamiltonian

$$-H_{N,\boldsymbol{\beta}}(\boldsymbol{\sigma}) = \sum_{p\geqslant 1} 2^{-p}\beta_p H_{N,2p}(\boldsymbol{\sigma}) + h\sum_{i\leqslant N}\sigma_i. \tag{2}$$

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Here, h represents the strength of an external field, and is fixed once and for all. It is understood that $H_{N,2p} = 0$ if $2p > N$. The term 2^{-p} is to ensure convergence and has no special meaning. We define the function

$$\xi_{\boldsymbol{\beta}}(x) = \sum_{p \geqslant 1} 2^{-2p} \beta_p^2 x^{2p}.$$

A simple computation shows that (if $\mathbb{E}$ denotes expectation in the r.v. $g_{i_1,\dots,i_p}$) we have, for two configurations $\boldsymbol{\sigma}^1$ and $\boldsymbol{\sigma}^2$ that

$$\left| \frac{1}{N} \mathbb{E}\big(H_{N,\boldsymbol{\beta}}(\boldsymbol{\sigma}^1) H_{N,\boldsymbol{\beta}}(\boldsymbol{\sigma}^2)\big) - \xi_{\boldsymbol{\beta}}(R_{1,2}) \right| \leqslant c(N, \boldsymbol{\beta}),$$

where $c(N, \boldsymbol{\beta}) \to 0$ as $N \to \infty$ uniformly over every set $\{\boldsymbol{\beta};\, \|\boldsymbol{\beta}\|_2 \leqslant C\}$ and where $R_{1,2} = R_{1,2}(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) = N^{-1} \sum_{i \leqslant N} \sigma_i^1 \sigma_i^2$ is the overlap of the configurations $\boldsymbol{\sigma}^1$ and $\boldsymbol{\sigma}^2$.

We observe that ξ is infinitely differentiable, that $\xi_{\boldsymbol{\beta}}(x) = \xi_{\boldsymbol{\beta}}(-x)$, that $\xi_{\boldsymbol{\beta}}$ is convex and that $\xi''_{\boldsymbol{\beta}}(x) > 0$ for $x > 0$. Thus, as is proved in [4] we can use Parisi's formula to compute $\lim_{N \to \infty} N^{-1} \mathbb{E} \log \sum_{\boldsymbol{\sigma}} \exp(-H_{N,\boldsymbol{\beta}}(\boldsymbol{\sigma}))$. This formula, that will be explained below, involves a "functional order parameter", and the purpose of this Note is to provide a rigorous interpretation of this parameter. Background on spin glass models can be found in [2] and [3].

2. Statement of results

Consider an integer $k \geqslant 1$ and the set $\mathcal{M}_k$ of all probability measures on $[0, 1]$ that are of the type

$$\mu = \frac{1}{k} \sum_{1 \leqslant \ell \leqslant k} \delta_{q_\ell}. \tag{3}$$

We assume without loss of generality that $q_1 \leqslant \cdots \leqslant q_k$ and we define $q_0 = 0$ and $q_{k+1} = 1$. Given a convex function ξ on $\mathbb{R}$, we consider the function $F_{k+1}(x) = \log \operatorname{ch}(x)$ and for $1 \leqslant \ell \leqslant k$ we define recursively

$$F_\ell(x) = \frac{1}{m_\ell} \log \mathbb{E} \exp m_\ell F_{\ell+1}\Big(x + g\sqrt{\xi'(q_{\ell+1}) - \xi'(q_\ell)}\Big),$$

where $m_\ell = \ell/k$ and where g is standard normal. We define

$$F_0(x) = \mathbb{E} F_1\big(x + g\sqrt{\xi'(q_1)}\big) \tag{4}$$

and $\theta(x) = x\xi'(x) - \xi(x)$. It is simple to check that the quantity

$$\mathcal{P}(\xi, \mu) = \log 2 + F_0(h) - \frac{1}{2}\theta(0) + \frac{1}{2} \int \theta(x)\, \mathrm{d}\mu(x)$$

depends only on μ and not on the representation (3). We provide the set of probability measures $\mathcal{M}$ on $[0, 1]$ with the weak* topology. The following extends an important result of Guerra [1], Theorem 1.

Proposition 2.1. *The map $\mu \mapsto \mathcal{P}(\xi, \mu)$ is uniformly continuous on $\bigcup_{k \geqslant 1} \mathcal{M}_k$ and consequently it has a continuous extension to $\mathcal{M}$.*

Of course we use the same notation $\mathcal{P}(\xi, \mu)$ to denote this extension.

Corollary 2.2. *There exists $\mu_\xi \in \mathcal{M}$ such that $\mathcal{P}(\xi, \mu_\xi) = \inf_\mu \mathcal{P}(\xi, \mu)$.*

The result of [4] can then be formulated as follows

Theorem 2.3. *For each $\boldsymbol{\beta}$ there is $\mu_{\boldsymbol{\beta}} \in \mathcal{M}$ such that*

$$\lim_{N\to\infty} \frac{1}{N} \mathbb{E} \log \sum_{\boldsymbol{\sigma}} \exp\bigl(-H_{N,\boldsymbol{\beta}}(\boldsymbol{\sigma})\bigr) = \mathcal{P}(\xi_{\boldsymbol{\beta}}, \mu_{\boldsymbol{\beta}}) = \inf_{\mu} \mathcal{P}(\xi_{\boldsymbol{\beta}}, \mu). \tag{5}$$

We will call a measure $\mu_{\boldsymbol{\beta}}$ that satisfies the last equality in (5) a *Parisi measure*. The physicists think to this object as a parameter which, when fixed at the appropriate value, allows the computation of the left-hand side of (5). The name "functional order parameter" arises from the identification of a probability measure μ with the function $x \mapsto \mu((0, x])$.

The following two conjectures seem closely related to each other.

Conjecture 2.4. *There is only one Parisi measure.*

In words, we conjecture that the function $\mu \mapsto \mathcal{P}(\xi_{\boldsymbol{\beta}}, \mu)$ attains its minimum at a unique point.

Conjecture 2.5. *The map $\boldsymbol{\beta} \mapsto \mathcal{P}(\xi_{\boldsymbol{\beta}}, \mu_{\boldsymbol{\beta}})$ is Gateaux differentiable at every point.*

Since the left-hand side of (5) is a convex function of $\boldsymbol{\beta}$, we also have

$$\text{The function } \boldsymbol{\beta} \mapsto \mathcal{P}(\xi_{\boldsymbol{\beta}}, \mu_{\boldsymbol{\beta}}) \text{ is convex.} \tag{6}$$

We do not know how to show this directly.

Definition 2.6. We say that a point $\boldsymbol{\beta}_0$ is *regular* if the map $\boldsymbol{\beta} \mapsto \mathcal{P}(\xi_{\boldsymbol{\beta}}, \mu_{\boldsymbol{\beta}})$ is Gateaux differentiable at this point.

Conjecture 2.5 means that we expect that every $\boldsymbol{\beta}$ is regular. It follows classically from (6) that the "generic" point $\boldsymbol{\beta}$ is regular. We now fix an integer $s \geqslant 1$ and $\boldsymbol{\beta}$. For $t \in \mathbb{R}$ we consider the point $\boldsymbol{\beta}(t)$ obtained by replacing the coordinate β_s by $\beta_s + t$. We write ξ_t instead of $\xi_{\boldsymbol{\beta}(t)}$.

Proposition 2.7. *For every measure $\mu \in \mathcal{M}$ the map $t \mapsto \mathcal{P}(\xi_t, \mu)$ is differentiable. Moreover, if μ is a Parisi measure $\mu_{\boldsymbol{\beta}}$, the value of the derivative at $t = 0$ is $2^{-2s}\beta_s(1 - \int x^{2s}\,\mathrm{d}\mu(x))$.*

Theorem 2.8. *If the point $\boldsymbol{\beta}$ is regular, then*

$$\forall s, \beta_s \neq 0 \Rightarrow \lim_{N\to\infty} \mathbb{E}\bigl\langle R_{1,2}^{2s}\bigr\rangle = \int x^{2s}\,\mathrm{d}\mu_{\boldsymbol{\beta}}(x), \tag{7}$$

where $\langle\cdot\rangle$ denotes an average on Σ_N^2 for the Gibbs measure with Hamiltonian (2).

As a consequence of (7) we can state that, provided that $\boldsymbol{\beta}$ is regular and $\beta_s \neq 0$ for each s, the limiting distribution of $R_{1,2}^2$ is the image of $\mu_{\boldsymbol{\beta}}$ under the map $x \mapsto x^2$. We observe that when $h = 0$ there is global symmetry around 0 so that the law of $R_{1,2}$ is symmetric around 0, so that μ cannot be the limiting distribution of $R_{1,2}$ unless it is concentrated at 0 (the high-temperature case). However we have the following.

Proposition 2.9. *If $h > 0$ then* $\lim_{N\to\infty} \mathbb{E}\langle 1_{\{R_{1,2}\leqslant 0\}}\rangle = 0$.

As a consequence of this result and of Theorem 2.8, we see that when $h > 0$, and provided that $\beta_s \neq 0$ for each s, the measure $\mu_{\boldsymbol{\beta}}$ is the limiting distribution of $R_{1,2}$. This provides the desired interpretation of the Parisi measure $\mu_{\boldsymbol{\beta}}$.

Proof of Theorem 2.8. Consider the function $f(t) = \mathcal{P}(\xi_t, \mu_{\boldsymbol{\beta}(t)})$. Since we assume that $\boldsymbol{\beta}$ is regular, this function is differentiable at $t = 0$. Since $\mu_{\boldsymbol{\beta}(t)}$ is a Parisi measure for ξ_t we have $f(t) \leqslant g(t) := \mathcal{P}(\xi_t, \mu_{\boldsymbol{\beta}})$. By Proposition 2.7, the function g is differentiable at $t = 0$, so that

$$f'(0) = g'(0) = 2^{-2s}\beta_s\left(1 - \int x^{2s}\,\mathrm{d}\mu_{\boldsymbol{\beta}}(x)\right). \tag{8}$$

Consider now the functions $f_N(t) = N^{-1}\mathbb{E}\log\sum_{\boldsymbol{\sigma}}\exp(-H_{N,\boldsymbol{\beta}(t)}(\boldsymbol{\sigma}))$. By a standard computation, we have

$$f_N'(t) = 2^{-2s}\beta_s\left(1 - \mathbb{E}\left\langle R_{1,2}^{2s}\right\rangle_t\right) + \varepsilon_N, \tag{9}$$

where ε_N goes to zero when $N \to \infty$, uniformly at t bounded, and where $\langle\cdot\rangle_t$ denotes an average on Σ_N^2 for the Gibbs measure with Hamiltonian $H_{N,\boldsymbol{\beta}(t)}$. Since the functions $t \mapsto f_N(t)$ are convex, and since their limit $f(t)$ is differentiable at $t = 0$, we have that $f'(0) = \lim_{N\to\infty} f_N'(0)$, and combining with (7) and (8) concludes the proof. □

One can hope that a simple underlying structure exists, that will obviate Conjectures 2.4 and 2.5. In the meantime however, the key point of the proofs of Propositions 2.1 and 2.7 is that, even though the definition of $F_0(x)$ in (4) involves a very large number of steps when r is large, the quantities of importance (such as F_0'') can be controlled independently of this number of steps, a fact that is already crystal clear in the formulation of Theorem 1 of [1], and that played a key role in the proofs of [4]. Proposition 2.9 is a consequence of the methods of [4].

One might wonder whether (as suggested by physical intuition) the measure $\mu_{\boldsymbol{\beta}}$ completely describes the asymptotic properties of the system. If this is the case, one should be able to compute all the relevant physical quantities of the system in function of it, and in particular quantities of the following type

$$\lim_{N\to\infty}\mathbb{E}\langle R_{1,2}R_{1,3}\rangle,$$
$$\lim_{N\to\infty}\mathbb{E}\langle R_{1,2}R_{3,4}\rangle = \lim_{N\to\infty}\mathbb{E}\langle R_{1,2}\rangle^2,$$

where we have introduced two more replicas $\boldsymbol{\sigma}^3$ and $\boldsymbol{\sigma}^4$, and where of course $R_{2,3} = N^{-1}\sum_{i\leqslant N}\sigma_i^2\sigma_i^3$. The author feels this is a very interesting research program.

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