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Differential Geometry

Ricci flow on compact Kähler manifolds of positive bisectional curvature

Huai-Dong Cao^{a,b,*}, Bing-Long Chen^{c,d}, Xi-Ping Zhu^{c,d}

^a Department of Mathematics, Texas A&M University, College Station, TX 77843-3368, USA

^b Institute for Pure and Applied Mathematics at UCLA, IPAM Building, 460 Portola Plaza, Box 957121, Los Angeles, CA 90095-7121, USA

^c Department of Mathematics, Zhongshan University, Guangzhou, 510275, PR China

^d The Institute of Mathematical Sciences, Unit 601, 6/F, Academic Building No. 1, The Chinese University of Hong Kong, Shatin, Hong Kong

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Abstract

This Note announces a new proof of the uniform estimate on the curvature of metric solutions to the Ricci flow on a compact Kähler manifold with positive bisectional curvature. This proof does not pre-suppose the existence of a Kähler–Einstein metric on the manifold, unlike the recent work of XiuXiong Chen and Gang Tian. It is based on the Harnack inequality for the Ricci–Kähler flow (see *Invent. Math.* 10 (1992) 247–263), and also on an estimation of the injectivity radius for the Ricci flow, obtained recently by Perelman. *To cite this article: H.-D. Cao et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).*

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Résumé

Le flot de Ricci sur une variété kählérienne compacte à courbure bisectionnelle positive. Cette Note annonce une nouvelle démonstration de l'estimée uniforme de la courbure des métriques solutions du flot de Ricci sur une variété kählérienne compacte à courbure bisectionnelle positive. La démonstration proposée ne suppose pas l'existence d'une métrique d'Einstein–Kähler sur la variété, contrairement à un travail récent de XiuXiong Chen et de Gang Tian. Elle s'appuie sur l'inégalité de Harnack pour le flot de Ricci–Kähler (voir *Invent. Math.* 10 (1992) 247–263), et aussi sur une estimation du rayon d'injectivité du flot de Ricci obtenue récemment par Perelman. *Pour citer cet article: H.-D. Cao et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).*

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1. The main result

We announce a proof of the uniform estimate on the curvature of solutions to the Ricci flow on a compact Kähler manifold M^n with positive bisectional curvature without using the existence of Kähler–Einstein metrics.

* Corresponding author.

E-mail addresses: hcao@ipam.ucla.edu, cao@math.tamu.edu (H.-D. Cao), blchen@math.cuhk.edu.hk (B.-L. Chen), stszxp@zsu.edu.cn (X.-P. Zhu).

Given a compact Kähler manifold M^n of complex dimension n and a Kähler metric $\tilde{g} = \sum \tilde{g}_{i\bar{j}}(x) dz^i dz^{\bar{j}}$ on M^n of positive holomorphic bisectional curvature, consider the normalized Kähler–Ricci flow, on M^n :

$$\frac{\partial}{\partial t} g_{i\bar{j}}(x, t) = -R_{i\bar{j}}(x, t) + g_{i\bar{j}}(x, t), \quad (1)$$

with initial condition $g_{i\bar{j}}(x, 0) = \tilde{g}_{i\bar{j}}$. By a result of Bishop–Goldberg [2], we can assume the Kähler class of the metric \tilde{g} satisfies the condition:

$$[\tilde{\omega}] = [\tilde{\Sigma}] = \pi c_1(M^n), \quad (2)$$

where $\tilde{\omega} = (\sqrt{-1}/2)\tilde{g}_{i\bar{j}} dz^i \wedge dz^{\bar{j}}$ and $\tilde{\Sigma} = (\sqrt{-1}/2)\tilde{R}_{i\bar{j}} dz^i \wedge dz^{\bar{j}}$ are the Kähler form, the Ricci form of the metric \tilde{g} respectively, while $c_1(M)$ denotes the first Chern class. Under the initial condition (2), the first author [3] (see also Proposition 1.1 in [4]) showed that the solution $g(x, t) = \sum g_{i\bar{j}}(x, t) dz^i dz^{\bar{j}}$ to the normalized flow (1) exists for all time. On the other hand, by the work of Mok [10] (and Bando [1] for $n = 3$), the solution metric $g(x, t)$ is known to have positive bisectional curvature for any $t > 0$. Our main result is on the uniform estimate of the curvature independent of t :

Theorem 1.1. *Let M^n be a compact Kähler manifold of complex dimension n and $\tilde{g} = \sum \tilde{g}_{i\bar{j}}(x) dz^i dz^{\bar{j}}$ be a Kähler metric on M^n of positive holomorphic bisectional curvature satisfying condition (2). Let $g(x, t)$ be the solution to the normalized Kähler–Ricci flow (1) on M^n with the initial condition $g(x, 0) = \tilde{g}$. Then $g(x, t)$ is nonsingular. Namely, the curvature of $g(x, t)$ is uniformly bounded independent of t .*

As a consequence of the Theorem 1.1. and a result of Perelman [12] (Proposition 11.2) we have:

Corollary 1.2. *Let $g(x, t)$ be the solution to the normalized flow as in the Theorem 1.1. Then there exists a subsequence $t_k \rightarrow \infty$ so that $g(x, t_k)$ converges to a gradient shrinking Kähler–Ricci soliton g_∞ .*

Remark 1. By the solution of the Frankel conjecture by Mori [11] and Siu–Yau [14], M^n is known to be biholomorphic to the complex projective space $\mathbb{C}P^n$, hence admits a Kähler–Einstein metric, the standard Fubini–Study metric on $\mathbb{C}P^n$. Therefore if we apply the result of Mori and Siu–Yau, the soliton metric g_∞ above is necessarily Kähler–Einstein.

Whether the solution g above is nonsingular or not has been one of the important open problems in the study of Hamilton’s Ricci flow on compact Kähler manifolds. For $n = 1$, the main theorem is proved by Richard Hamilton [8], based on his Harnack inequality and entropy estimate for the Gaussian curvature (see also the further improvements by Chow [7] and Hamilton [9]). In [5,6], Chen–Tian obtained the uniform estimate on the curvature of $g(x, t)$, but only assuming in addition the existence of the Kähler–Einstein metric (a consequence of Mori [11] and Siu–Yau [14]), so that a Moser–Trudinger type inequality of Tian [15], valid for Kähler–Einstein manifolds, can be applied. Chen–Tian also proved the solution to the normalized flow converges to an Kähler–Einstein metric. In contrast, our proof of the main theorem does not rely on the existence of Kähler–Einstein metrics, but instead on first author’s Harnack inequality for the scalar curvature $R(x, t)$ which is a consequence of the Li–Yau–Hamilton estimate for the Kähler–Ricci flow [4], and a very recent local injectivity radius estimate of Perelman for the Ricci flow [12]. In next section, we shall sketch the proof of the Theorem 1.1. Complete details of the proof and discussions about convergence of $g(x, t)$ will appear elsewhere.

Remark 2. Our proof also works for compact Kähler manifolds with nonnegative bisectional curvature.

2. Outline of the proof

Let $g(x, t)$ be the solution to the normalized Kähler–Ricci flow (1) on compact Kähler manifold M^n with positive bisectional curvature whose initial metric satisfies condition (2). It is easy to see that under flow (1), the volume $V = V(t)$, the total scalar curvature $\int_{M^n} R(x, t) \, dv_t$, and the average scalar curvature

$$r = r(t) = \frac{1}{V} \int_{M^n} R(x, t) \, dv_t$$

of $(M^n, g(x, t))$ are all constant in t . In fact, $r(t) = n$ for all t . Clearly, it suffices to show that the scalar curvature of g is uniformly bounded from above in t :

$$R(x, t) < C, \quad \forall x \in M^n \, \forall t \in [0, \infty)$$

for some constant $C > 0$ independent of t . To do this, let us first recall:

The Harnack inequality for the scalar curvature (Cao [4]). *For any $x, y \in M^n$ and $0 < t_1 < t_2 < \infty$, the scalar curvature $R(x, t)$ of solution metric $g(x, t)$ satisfies the inequality:*

$$R(x, t_1) \leq R(y, t_2) \frac{e^{t_2} - 1}{e^{t_1} - 1} e^{\Delta/4}. \tag{3}$$

Here $\Delta = \Delta(x, t_1; y, t_2) = \inf_{\gamma} \int_{t_1}^{t_2} |\gamma'(\tau)|^2 \, d\tau$, where the infimum is taken over all space–time curves from (x, t_1) to (y, t_2) , and $|\gamma'(\tau)|$ is the velocity of γ at time $\tau \in [t_1, t_2]$.

For any $x \in M^n$ and $t > 1$, set $t_1 = t$ and $t_2 = t + 1$. We can find $y \in M^n$ such that $R(y, t + 1) = r(t + 1) = n$. It then follows from the Harnack inequality (3),

$$\begin{aligned} R(x, t) &\leq R(y, t + 1) \frac{e^{t+1} - 1}{e^t - 1} e^{\Delta(x,t;y,t+1)/4} \\ &\leq n(e + 1) e^{\Delta(x,t;y,t+1)/4}. \end{aligned} \tag{4}$$

On the other hand, from (1) one can show that

$$\Delta(x, t; y, t + 1) \leq e^2 d^2(x, y; t), \tag{5}$$

where $d(x, y; t)$ denotes the geodesic distance between x and y with respect to the metric $g_{i\bar{j}}(t)$. In particular, by (4) and (5),

$$R(z, t) \leq n(e + 1) \exp\left(\frac{1}{4} e^2\right)$$

for all $z \in B_t(y, 1)$, the geodesic ball centered at y of radius 1 with respect to the metric $g_{i\bar{j}}(t)$. Then by Perelman’s no local collapsing result (Theorem 4.1 or its corollary in [12]), there exists a constant $\beta > 0$ independent of t such that the volume of the geodesic ball $B_t(y, 1)$ has a uniform lower bound:

$$\text{Vol}(B_t(y, 1)) \geq \beta. \tag{6}$$

From (6) one can, by a volume comparison argument of Yau (cf. [13]), deduce a uniform upper bound on the diameter d_t of $(M^n, g(x, t))$: there is a constant $D = D(\beta) > 0$ independent of t , such that for all $t > 1$,

$$d_t \leq D. \tag{7}$$

Now (4), (5), and (7) imply that, for any $t > 1$ and any $x \in M^n$,

$$R(x, t) \leq n(e + 1) \exp(e^2 D^2/4),$$

which is our desired uniform estimate.

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