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Duality of metric entropy in Euclidean space

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Abstract

Let K be a convex body in a Euclidean space, K° its polar body and D the Euclidean unit ball. We prove that the covering numbers $N(K, tD)$ and $N(D, tK^\circ)$ are comparable in the appropriate sense, uniformly over symmetric convex bodies K, over $t > 0$ and over the dimension of the space. In particular this verifies the duality conjecture for entropy numbers of linear operators, posed by Pietsch in 1972, in the central case when either the domain or the range of the operator is a Hilbert space. ***To cite this article: S. Artstein et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Dualité d'entropie métrique. Soit K un corps convexe d'un espace euclidien. Nous notons K° le polaire de K et D la boule unité euclidienne. Nous montrons que les nombres de recouvrement $N(K, tD)$ et $N(D, tK^\circ)$ sont équivalents dans un sens approprié, uniformément sur tous les corps convexes symétriques, pour $t > 0$ et pour toute dimension d'espace. En particulier, nous confirmons la conjecture concernant la dualité des nombres d'entropie des opérateurs compacts entre espaces de Banach, conjecture formulée par Pietsch en 1972 dans le cas fondamental où l'un des espaces est hilbertien. ***Pour citer cet article : S. Artstein et al., C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

For two convex bodies K and T in $\mathbb{R}^n$, the covering number of K by T, denoted $N(K, T)$, is defined as the minimal number of translates of T needed to cover K,

$$N(K, T) = \min\left\{N\colon \exists x_1, \ldots, x_N \in \mathbb{R}^n,\ K \subset \bigcup_{i \leqslant N} x_i + T\right\}.$$

We denote by D the Euclidean unit ball in $\mathbb{R}^n$. In this paper we announce and sketch the proof of the following duality result for covering numbers. (The details will appear elsewhere.)

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Theorem 1. *There exist two universal constants α and β such that for any dimension n, any convex body $K \subset \mathbb{R}^n$ symmetric with respect to the origin and any $t > 0$, one has*

$$N(D, \alpha t K^\circ)^{1/\beta} \leqslant N(K, tD) \leqslant N(D, \alpha^{-1} t K^\circ)^{\beta}, \tag{1}$$

where $K^\circ := \{u \in \mathbb{R}^n: \sup_{x \in K} \langle x, u\rangle \leqslant 1\}$ is the polar body of K.

The best constant β that our approach yields is $\beta = 2 + \varepsilon$ for any $\varepsilon > 0$, with $\alpha = \alpha(\varepsilon)$ (i.e., with α depending on ε and only on ε).

Theorem 1 resolves an old problem, going back to Pietsch ([6], p. 38) and referred to as the "duality conjecture for entropy numbers", in a special yet most important case. The problem can be stated in terms of covering numbers in the following way (below and in what follows we shall abbreviate "symmetric with respect to the origin" to just "symmetric").

Conjecture 2. *Do there exist two numerical constants $a, b \geqslant 1$ such that for any dimension n, and for any two symmetric convex bodies K and T in $\mathbb{R}^n$ one has*:

$$\log N(T^\circ, aK^\circ) \leqslant b \log N(K, T), \tag{2}$$

where A° denotes the polar body of A?

Theorem 1 satisfies this conjecture in the case where one of the two bodies is a Euclidean ball or, more generally, by affine invariance of the problem, when one of the two bodies is an ellipsoid. (Note that the additional parameter t presented in the statement of Theorem 1 does not affect the generality.) In the special case where *both* bodies are ellipsoids it is well known and easy to check that there is equality in (2), with $a = b = 1$.

The quantity $\log N(K, tT)$ has a clear information-theoretic interpretation: it is the complexity of K (measured in bits if the logarithm is to the base 2) at the level of resolution t with respect to the metric associated with T (e.g., Euclidean if $T = D$). Accordingly, (1) means that the complexity of K in the Euclidean sense is controlled by that of the Euclidean ball with respect to $\|\cdot\|_{K^\circ}$ (the gauge of K°, i.e., the norm whose unit ball is K°), and vice versa, at *every* level of resolution. While it is clear that these two complexities should be related, the universality of the link that we establish is somewhat surprising. The covering numbers appear also in many other areas of mathematics, for example both quantities $N(K, tD)$ and $N(D, tK^\circ)$ enter the theory of Gaussian processes (see, e.g., the survey [4] and its references).

Conjecture 2 originated in operator theory, and so we restate it below in the language of entropy numbers of operators. For two Banach spaces X and Y, with unit balls $B(X)$ and $B(Y)$ respectively, and for a linear operator $u: X \to Y$, the k-th entropy number of u is defined by:

$$e_k(u) := \inf\bigl\{\varepsilon:\ N\bigl(uB(X), \varepsilon B(Y)\bigr) \leqslant 2^{k-1}\bigr\}.$$

Considering the entropy numbers of the dual operator $u^*: Y^* \to X^*$ means covering $(B(Y))^\circ$ with (translates of) $\varepsilon(B(X))^\circ$. One sees easily that $e_1(u) = \|u\|_{op}$ (the operator norm), and so $e_1(u) = e_1(u^*)$; similarly $e_k(u) \to 0$ (as $k \to \infty$) iff $e_k(u^*) \to 0$ (iff u is a compact operator). As the sequence $(e_k(\cdot))$ quantifies the compactness of an operator, it is natural to ask to what extent $(e_k(u))$ and $(e_k(u^*))$ behave similarly. This is the context in which the original duality conjecture was formulated, and it read as follows.

Conjecture 2A (Duality Conjecture in the language of entropy numbers). *Do there exist numerical constants $a, b \geqslant 1$, such that for any two Banach spaces X and Y, any linear operator $u: X \to Y$ and any natural number k, one has*:

$$e_{bk}(u^*) \leqslant a e_k(u)?$$

Our Theorem 1 verifies thus the duality conjecture (when expressed in terms of entropy numbers) in the case when one of the two spaces, either the domain or the range of the operator u, is a Hilbert space. (The passage from finite- to infinite-dimensional setting is achieved by a straightforward approximation once dimension-free estimates are available.) Some special cases of the problem have been studied before, and some particular results were established, see [2] for references. The known results which we need for our proof are those from [1,3,5] and [7].

Our proof consists of three main steps. The first is a duality result involving the diameter of the set. In fact we have two alternative first steps, the first of which was shown in [2]: denoting $\psi(x) = C_0(\log x)^3$ for some appropriate numerical constant $C_0 > 0$, we have:

Lemma 3. *If $K \subset RD \subset \mathbb{R}^n$ is a convex symmetric body, then*

$$N\big(D, \psi(R)K^\circ\big) \leqslant N(K, D)^2, \tag{3}$$

$$N(K, D) \leqslant N\big(D, \big(1/\psi(R)\big)K^\circ\big)^3. \tag{4}$$

The other alternative for the first step is the following fact, the proof of which is significantly easier than the proof of Lemma 3. However, there is a price to pay for simplifying the argument: instead of $\beta = 2 + \varepsilon$ in the exponent in (1), we get some other, perhaps huge, universal constant.

Lemma 4. *There exist universal constants c_3 and C_4 such that for any dimension n, and for any symmetric convex body $K \subset RD \subset \mathbb{R}^n$, we have: $N(K, D) \leqslant N(D, c_3K^\circ)^{C_4R^2}$ and $N(D, K^\circ) \leqslant N(K, c_3D)^{C_4R^2}$.*

With Lemma 4 (or 3) at hand, we come to the second step which is an iteration procedure. At each step of the iteration we truncate the body at a certain radius, and use elementary geometric inequalities for covering numbers. (In the dual inequality at each step we add a small ball to the body.) We thus bound a covering number by a product of such numbers, where for each factor in the product we have control on the diameter of the respective body.

Lemma 5. *For any symmetric convex body $K \subset \mathbb{R}^n$ and any sequence $R_0 < R_1 < \cdots < R_s$,*

$$N(D, R_0K^\circ) \leqslant N(D, R_sK^\circ)\prod_{j=0}^{s-1} N\left(D, \frac{R_j}{2}(K \cap R_{j+1}D)^\circ\right), \tag{5}$$

$$N(K, R_0D) \leqslant N(K, R_sD)\prod_{j=0}^{s-1} N(2K \cap R_{j+1}D, R_jD). \tag{6}$$

We choose the sequence R_j to be a geometric progression of the form $R_j = C_1C_2^j$, for big enough (but universal) constants C_1 and C_2. Lemmas 4 and 5 imply now the following:

Corollary 6. *With the above choice of the sequence (R_j) we have, for every symmetric convex body K,*

$$N(D, R_0K^\circ) \leqslant N(D, R_sK^\circ)\prod_{j=0}^{s-1} N\left(K \cap R_{j+1}D, \frac{c_3}{2}R_jD\right)^{4C_4C_2^2}, \tag{7}$$

$$N(K, R_0D) \leqslant N(K, R_sD)\prod_{j=0}^{s-1} N\big(D, c_3R_j(2K \cap R_{j+1}D)^\circ\big)^{C_4C_2^2}. \tag{8}$$

If we use Lemma 3 instead of Lemma 4 and an appropriate sequence R_j, we can prove a similar corollary with better constants in the exponent, 2 and 3 respectively, which – with still more work – can be improved to $1 + \varepsilon$ in both inequalities.

In the third and last step we collapse the long products of covering numbers appearing in (7) and (8) to products consisting of just several terms. The largest R_s will be chosen to exceed the diameter of the set K, and so the terms

$N(K, R_s D)$ and $N(D, R_s K^\circ)$ will both equal to 1. We use the following two super-multiplicativity properties of covering numbers.

Lemma 7. *Let $R > r > 5S > 5s$. Then*

$$N(K \cap RD, rD)N(K \cap SD, sD) \leqslant N\left(K \cap RD, \frac{s}{4}D\right), \tag{9}$$

$$N\big(D, r(K \cap RD)^\circ\big)N\big(D, s(K \cap SD)^\circ\big) \leqslant N\left(D, \frac{s}{4}(K \cap RD)^\circ\right). \tag{10}$$

We successively apply Lemma 7 to the long products in (7) and (8). However, since for two neighboring factors in the products the hypothesis of Lemma 7 does not hold, an additional trick is required. We split the product into two parts, by grouping separately the factors corresponding to the odd and the even j's. If the growth of (R_j) (which depends on the choice of C_2) is fast enough, the conditions of Lemma 7 are satisfied for each two consecutive odd factors, and for each two consecutive even factors. We thus arrive at

$$N(D, C_1 K^\circ) \leqslant N(K, D)^{C_5} \quad \text{and} \quad N(K, C_1 D) \leqslant N(D, K^\circ)^{C_5},$$

where $C_5 = 8C_4C_2^2$ and C_1 are universal constants, which clearly yields (1). As indicated earlier, a (much) more careful analysis leads to a better constant $2 + \varepsilon$ in the exponent, with $C_1 = C_1(\varepsilon)$.

In fact, the last two steps of the proof give rise to the following proposition, valid for general symmetric convex bodies. It identifies the point in our scheme that is needed for generalizations beyond the Euclidean case. (Our decision to include this statement in the form below was influenced by discussions with Nicole Tomczak-Jaegermann.)

Proposition 8. *Let T be a convex symmetric body in a Euclidean space such that, for some constants $c, C > 0$, the following holds: if K is a convex symmetric body with $K \subset 4T$, then*

$$N(K, T) \leqslant N(T^\circ, cK^\circ)^C.$$

Then, for some other constants $c', C' > 0$ (depending only on c, C) and **any** *convex symmetric body K*

$$N(K, T) \leqslant N(T^\circ, c'K^\circ)^{C'}.$$

Dually, if K is fixed and the hypothesis holds for all T's verifying $K \subset 4T$, then the assertion holds for **any** *T.*

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