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# Equivariant cosymmetry and front solutions of the Dubreil–Jacotin–Long equation. Part 2: Exact solutions

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**Abstract**

The paper concerns existence of exact solutions of the Dubreil–Jacotin–Long equation describing large amplitude internal fronts in a continuously stratified fluid. The proof uses cosymmetric variant of the implicit function theorem based on the group invariance of the variational functional for DJL operator. Supercritical branching occurs near approximate front solutions at the boundary of continuous spectrum of the problem linearized with respect to the basic uniform flow. ***To cite this article: N. Makarenko, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***


**Résumé**

**Cosymétrie équivariante et solutions « fronts » de l'équation de Dubreil–Jacotin–Long. 2ème partie : Solutions exactes.** Cette Note traite de l'existence de solutions exactes de l'équation de Dubreil–Jacotin–Long (DJL), qui décrivent les fronts internes de grande amplitude dans un fluide continûment stratifié. La démonstration utilise une variante du théorème des fonctions implicites en présence d'une cosymétrie, basée sur le groupe d'invariance de la fonctionnelle variationnelle de l'opérateur de DJL. Une bifurcation supercritique a lieu au bord du spectre continu du problème linéarisé au voisinage de l'écoulement primaire. ***Pour citer cet article : N. Makarenko, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***


## 1. Introduction

In this Note we prove the existence of exact front solutions of DJL equation in weighted Sobolev spaces near the approximate solutions constructed in the Part 1 by scaling with small Boussinesq parameter. We apply here the modified Lyapunov–Schmidt procedure which is similar to the method suggested in the paper Beale [1] for the problem on surface solitary wave. The cosymmetry arguments are used to satisfy the orthogonality condition which appears due to the dissymmetry of the front-like flow. We refer to formulae (n) from the first part of the paper by notation I(n) everywhere in the text.

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## 2. Nonlinear maps associated with DJL operator

We consider the front problem for DJL equation in the strip $\Omega = \mathbb{R} \times (0, \pi)$

$$F(v;\sigma,\lambda) \stackrel{\text{def}}{=} \sigma(\rho v_x)_x + (\rho v_y)_y - \lambda\rho' v - f(v, \nabla v, y; \sigma) = 0, \quad (x, y) \in \Omega, \tag{1}$$

$$\text{(a) } v(x,0) = v(x,\pi) = 0; \qquad \text{(b) } v \to v^{\pm}, \quad \nabla v \to \nabla v^{\pm} \quad (x \to \pm\infty), \tag{2}$$

where the density profile $\rho = \rho(y + v, \sigma)$, $\rho' = \sigma^{-1}\rho_y(y + v, \sigma)$ satisfies the Condition A of the Part 1, nonlinearity $f$ has the form $f = (1/2)(\sigma^2 v_x^2 + \sigma v_y^2)\rho'$, and $\sigma > 0$ is the Boussinesq parameter. We look for $v = u_0 + \sigma u$ with a leading-order term $u_0(x, y; \sigma) = b_0^{-1} a_0(x) v^+(y; \sigma)$ where $a_0$ is the front solution of Eq. I(6), and the conjugate flow $v^+$ is given by Theorem I 3.1. Note that $u_0$ satisfies *exact* condition (2b) at infinity. We rewrite equation for the function $u$ as follows

$$\sigma u_{xx} + u_{yy} + u = \varphi(u, \sigma), \tag{3}$$

where the operator $\varphi$ collects nonlinear terms of Eq. (1). Now we introduce the function spaces adapted to the front problem. Let $L_2^{\alpha,\beta}(\mathbb{R})$ be the weighted Hilbert space of the functions $u(x)$ having finite norm

$$\|u\|_{\alpha,\beta}^2 = \int_{-\infty}^{+\infty} \left(\mathrm{e}^{-2\alpha x} + \mathrm{e}^{2\beta x}\right) \left|u(x)\right|^2 \mathrm{d}x,$$

with the exponents $0 < \alpha < \alpha_0$ and $0 < \beta < \beta_0$ where $\alpha_0, \beta_0$ are decay exponents of approximate solution. Further, we introduce the weighted Sobolev space $H_{\alpha,\beta}^k(\mathbb{R})$ of the functions $u(x)$ which have generalized derivatives up to the order $\leqslant k$ belonging to the class $L_2^{\alpha,\beta}(\mathbb{R})$. Finally, we define the class

$$X^k = \left\{u(x, y) \mid D_x^m D_y^n u \in L_2\left([0, \pi]; H_{\alpha,\beta}^k(\mathbb{R})\right) \ (0 \leqslant m + n \leqslant 2);\ u(\cdot, 0) = u(\cdot, \pi) = 0\right\}$$

and the class $Y^k = L_2([0, \pi]; H_{\alpha,\beta}^k(\mathbb{R}))$. Weighted Sobolev norms in $X^k$ and $Y^k$ will be denoted as $\|\cdot\|_k$ and $|\cdot|_k$ consequently. It is easy to verify that $\varphi(0, \sigma) = -\sigma^{-1} F(u_0; \sigma, \lambda_1^+(\sigma)) \in Y^k$ due to the exponential behaviour of $a_0$ for large $|x|$. Let $B_r = \{\|u\|_k < r\}$ be the open ball in the class $X^k$. For every $r > 0$ there exists $\sigma_1(r)$ such that the map $\varphi : B_r \times (0, \sigma_1) \to Y^k$ is a smooth map. We will use some properties of the Fréchet derivative $\varphi_u'$ of the operator $\varphi$ and its remainder $\tilde{\varphi}$ in the Taylor expansion

$$\varphi(u, \sigma) = \varphi(0, \sigma) + \varphi_u'(0, \sigma)u + \tilde{\varphi}(u, \sigma).$$

Introduce the element $\varphi_0 = f_1(a_0 \sin y) - b_0^{-1} a_0 f_1(b_0 \sin y) \in Y^k$ defined by the function $f_1$ from I(5). In addition, let $M : X^k \to Y^k$ be the linear differential operator acting by the formula $Mu = -(\psi_0 u_y)_y + \theta u$ where the coefficients are $\psi_0 = y + v_0$ and $\theta = \rho_0'(\psi_0) + (1 + \rho_0''(\psi_0))v_0 + \Lambda_1(b_0)$.

**Lemma 2.1.** *For $u, v \in B_r \subset X^k (k \leqslant l - 2)$ and $\sigma \in (0, \sigma_1)$ the estimates*

(i) $|\varphi(0, \sigma) - \varphi_0|_k \leqslant C\sigma$,
(ii) $|\varphi_u'(0, \sigma)u - \sigma M u|_k \leqslant C\sigma^2 \|u\|_k$,
(iii) $|\tilde{\varphi}(u, \sigma) - \tilde{\varphi}(v, \sigma)|_k \leqslant C\sigma^2 \|u - v\|_k$

*are valid with the constant $C$ independent on $\sigma$.*

## 3. Existence of exact front solutions

The form of a leading-order solution term motivate us to consider the direct sums of closed subspaces $X^k = \mathcal{X}^k \oplus QX^k$ and $Y^k = \mathcal{Y}^k \oplus QY^k$ with infinite-dimensional subspaces $\mathcal{X}^k = \{a(x)\sin y \mid a \in H^{k+2}_{\alpha,\beta}(\mathbb{R})\}$ and $\mathcal{Y}^k = \mathcal{X}^{k-2}$. The projection $Q$ can be chosen $Qu(x,y) = u(x,y) - u_1(x)\sin y$ where $u_1$ is the first Fourier coefficient for $u(\cdot, y)$ with respect to the basis $\{\sin ny\}$. Looking for the solution $u(x,y) = a(x)\sin y + w(x,y)$ with $w \in QX^k$ and $a \in H^{k+2}_{\alpha,\beta}(\mathbb{R})$ we obtain from (3) the system of equations

$$\begin{cases} \text{(a) } \sigma w_{xx} + w_{yy} + w = Q\varphi(a\sin y + w, \sigma), \\ \text{(b) } a'' + p''(a_0)a = \varphi^*(a\sin y + w, \sigma), \end{cases} \tag{4}$$

where $\varphi^*(u,\sigma) = (\sigma^{-1}\varphi(u,\sigma) - M(I-Q)u)_1$, brackets $(\varphi)_1$ mean the first Fourier coefficient of $\varphi$. Consider in more details the resolvents for the linear operators of the system (4). First of them, the operator $W : QY^k \to QX^k$, acts by the formula

$$Wf(x,y) = \sum_{n=2}^{\infty} w_n(x)\sin ny,$$

where the coefficients $w_n$ have the Fourier transforms $\widehat{w_n}(\xi) = -\hat{f}_n(\xi)/(\sigma\xi^2 + n^2 - 1)$. Functions $u \in L_2^{\alpha,\beta}(\mathbb{R})$ have the analytic Fourier transforms $\hat{u}(\xi + \mathrm{i}\eta)$ which belong to the Hardy class $H^2$ for the complex strip $-\alpha < \eta < \beta$. Therefore one can use the equivalent Hardy norm $\|u\|^2_{\alpha,\beta} = \|\hat{u}(\xi - \mathrm{i}\alpha)\|^2_{L_2(\mathbb{R})} + \|\hat{u}(\xi + \mathrm{i}\beta)\|^2_{L_2(\mathbb{R})}$ by the estimating. Since $\widehat{w_n}(\zeta)(n \geqslant 2)$ are analytic for $-\alpha < \operatorname{Im}\zeta < \beta$, we obtain following estimates.

**Lemma 3.1.** *For $0 \leqslant m + n \leqslant 2$ and $0 < \sigma \leqslant \sigma_2$ with $\sigma_2 < 3/\max(\alpha^2, \beta^2)$ the inequalities $\sigma^{m/2}|D_x^m D_y^n Wf|_k \leqslant C|f|_k$ are satisfied with the constant $C$ independent on $\sigma$. In addition, the operator $WQM : X^k \to QX^k$ $(k \leqslant l-2)$ is bounded uniformly in $\sigma$: $\|WQMu\|_k \leqslant C\|u\|_k$.*

The ordinary differential operator $D_x^2 + p''(a_0)I : H^{k+2}_{\alpha,\beta}(\mathbb{R}) \to H^k_{\alpha,\beta}(\mathbb{R})$ appearing in (4) is the Fréchet derivative of the nonlinear operator I(6) with respect to the front solution $a_0$. Consider integral operator

$$Kf(x) = a_1(x)\int_0^x a_2(x')f(x')\,\mathrm{d}x' + a_2(x)\int_x^{+\infty} a_1(x')f(x')\,\mathrm{d}x',$$

where $a_1(x) = a_0'(x)$, $a_2(x) = a_1(x)\int_0^x a_1^{-2}(x')\,\mathrm{d}x'$, and also $a'' + p''(a_0)a = -f$ if $a = Kf$.

**Lemma 3.2.** *If the function $f \in H^k_{\alpha,\beta}(\mathbb{R})$ satisfies the orthogonality condition $(a_1, f)_{L_2(\mathbb{R})} = 0$ then $Kf \in H^{k+2}_{\alpha,\beta}(\mathbb{R})$ is valid with the estimate $\|Kf\|_{H^{k+2}_{\alpha,\beta}(\mathbb{R})} \leqslant C\|f\|_{H^k_{\alpha,\beta}(\mathbb{R})}$.*

In accordance with this lemma the Fréchet derivative of DJL operator linearized about the approximate front solution is the Fredholm operator. It has one-dimensional kernel in $X^k$ spanned by the function $v_{0x} = a_0'(x)\sin y$, and one-dimensional defect space in $Y^k$. Applying the projectors $P = \|a_1\|^{-2}_{L_2(\mathbb{R})}a_1(a_1,\cdot)_{L_2(\mathbb{R})}$ and $I - P$ to Eq. (4b) and using the resolvents $W$ and $K$ we reduce the system (4) to equivalent system for $u \in X^k$ and parameter $c \in \mathbb{R}$:

$$\text{(a) } u = cv_{0x} + U(u,\sigma), \qquad \text{(b) } \omega(u,\sigma) = 0. \tag{5}$$

The nonlinear mapping $U$ is defined here by the formula

$$U(u,\sigma) = WQ\varphi(u,\sigma) - K(I-P)(I-Q)\big(\sigma^{-1}\varphi(u,\sigma) - Mu + MWQ\varphi(u,\sigma)\big),$$

the functional $\omega$ has the form $\omega(u,\sigma) = (\sigma^{-1}\varphi(u,\sigma) - M(I-Q)u, v_{0x})_{L_2(\Omega)}$. Let $\sigma_3 = \min(\sigma_1, \sigma_2)$, then the map $U : B_r \times (0, \sigma_3) \to X^k (k \leqslant l-3)$ is bounded uniformly in $\sigma$. Here the bound depends on the $Y^{l-1}$-norm of element $\varphi_0$ being sufficiently smooth, and the $\mathrm{O}(\sigma^{-1})$ term of $U$ vanishes at leading order due to orthogonality $(\varphi_0, \sin y)_{L_2[0,\pi]} = 0$. Moreover, this operator has the Lipschitz constant of the order $\mathrm{O}(\sigma)$ due to Lemmas 2.1 and 3.1, 3.2. Therefore for small $\sigma$ there exists the solution family $u(c,\sigma)$ of Eq. (5a).

Finally we consider the bifurcation equation (5b). Note that similar orthogonality condition for a symmetric solitary waves can be satisfied by the restriction to the classes of even functions. The restriction is impossible *a priori* for the front problem since internal bores are not symmetric flows in general case. We use the potentiality and group invariance of the Dubreil–Jacotin–Long operator in order to remove this obstacle. Namely, the divergence relation I(3) implies the identity

$$\big(F\big(v; \sigma, \lambda_1^+(\sigma)\big), v_x\big)_{L_2(\Omega)} = 0 \tag{6}$$

which is satisfied for all $v = u_0 + \sigma u$ with arbitrary $u \in B_r$. The operator $X = \partial_x$ is here the cosymmetry for DJL equation. We rename it *equivariant cosymmetry* since this operator is generated obviously by the $x$-autonomy of Lagrangian $L$. More generally [2], if a potential operator $F$ has invariant Lagrangian under action of Lie group $G$, then cosymmetry is given by the Lie algebra of infinitesimal generators of group $G$ factorized with respect to isotropy subgroup of basic solution $u_0$. Yudovich [3,4] explained by the presence of cosymmetry the branching of solution families near a known non-cosymmetric solution $u_0 (Xu_0 \neq 0)$. In the case under consideration, parametric solution $u(c,\sigma)$ appears since the projection of $F(u_0 + \sigma u(c,\sigma), \sigma)$ onto the defect space vanishes due to (6) identically in $c$, so Eq. (5b) is fulfilled automatically.

The following proposition is the main result of the paper.

**Theorem 3.3.** *Let the density profile satisfy Condition* A *of the Part* 1. *Then for each supercritical conjugate flows of the first mode given by Theorem* I 3.1 *and satisfying Condition* B (*Part* 1) *there exists exact solution of the problem* (1), (2) *having the form* $v = b_0^{-1} a_0(x) v^+(y;\sigma) + \sigma u(x,y;\sigma)$, *where* $u \in X^{l-3}$ *is uniformly bounded as* $\sigma \to 0$.

Any given number of supercritical fronts of the first mode are obtained for the densities $\rho$ such that the function $\Lambda_1(b)$ has the sequence of monotonically decreasing minima $\Lambda_1(b_0) > \Lambda_1(b_1) > \cdots > \Lambda_1(b_m)$ located at the points $0 < b_0 < b_1 < \cdots < b_m < 1$, or the sequence of monotonically increasing minima $\Lambda_1(b_0) < \Lambda_1(b_1) < \cdots < \Lambda_1(b_m)$ in the case $-1 < b_0 < b_1 < \cdots < b_m < 0$.

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