Probability Theory

Gross–Sobolev spaces on path manifolds: uniqueness and intertwining by \( \text{Itô maps} \)

K. David Elworthy \(^a\), Xue-Mei Li \(^b\)

\(^a\) Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK
\( ^b\) The Department of Computing and Mathematics, The Nottingham Trent University, Nottingham NG7 1AS, UK

Received 1 October 2003; accepted 6 October 2003

Presented by Jean-Michel Bismut

Abstract

Conditions are given under which the solution map \( I \) of a stochastic differential equation on a Riemannian manifold \( M \) intertwines the differentiation operator \( d \) on the path space of \( M \) and that of the canonical Wiener space, \( d_Q^* I^* = I^* d_{C_{x_0}^*} M \). A uniqueness property of \( d \) on the path space follows. Results are also given for higher derivatives and covariant derivatives. To cite this article: K.D. Elworthy, X.-M. Li, C. R. Acad. Sci. Paris, Ser. I 337 (2003).

1. Introduction

Let \( M \) be a compact \( C^\infty \) Riemannian manifold with Levi-Civita connection \( \nabla \). For \( x_0 \) a point in \( M \) and \( T > 0 \), let \( C_{x_0}^\infty \) be the \( C^\infty \) manifold of continuous paths \( \sigma : [0, T] \to M \) with \( \sigma(0) = x_0 \), equipped with Brownian motion measure \( \mu_{x_0} \). Its tangent space \( T_\sigma C_{x_0}^\infty \) is the space of continuous vector fields on \( M \) along \( \sigma \) vanishing at 0. Let \( //_t(\sigma) : T_{x_0} C_{x_0}^\infty \to T_{\sigma(t)} M \) be the stochastic parallel translation along \( \sigma \) defined almost surely. Denote by \( W_t \equiv W_t(\sigma) : T_{x_0} M \to T_{\sigma(t)} M \) the damped parallel translation along \( \sigma \) defined by

\[
\frac{d}{dt} //^{-1}_t(\sigma) W_t(v) = -\frac{1}{2} //^{-1}_t(\sigma) \text{Ric}^g_{x_0}(W_t(v)), \quad W_0(v) = v, \quad v \in T_{x_0} M
\]
for $\text{Ric}_{\sigma}^{\sigma} : T_{x}M \to T_{x}M$ given by the Ricci curvature. Set $\langle U, V \rangle_{\sigma} := \int_{0}^{T} \langle W_{t} \frac{\partial}{\partial t} W_{t}^{-1}(U_{t}), W_{t} \frac{\partial}{\partial t} W_{t}^{-1}(V_{t})\rangle_{\sigma} \, dt$. Let $\mathcal{H}_{\sigma}$ be the Hilbert space given for almost all $\sigma$ by $\mathcal{H}_{\sigma} = \{ V \in T_{x}C_{x_{0}}M \mid W_{t}^{-1}(V) \text{ is absolutely continuous, } V_{0} = 0 \text{ and } \| V \|_{2}^{2} < \infty \}$ with the inner product $\langle ., . \rangle_{\sigma}$. Note that $\mathcal{H}_{\sigma}$ is the same as the Bismut tangent space at $\sigma$, as used for example in [5], apart from the choice of inner product; and in the compact manifold and Brownian motion case considered here we could equally use either inner product (though Corollary 2.3(b) would need modification).

Choose a linear subspace $\text{Dom}(d_{H})$ of $L^{2}(C_{x_{0}}M; \mathbb{R})$ such that

(i) $\text{Dom}(d_{H})$ contains smooth cylindrical functions on $C_{x_{0}}M$

(ii) Each $f \in \text{Dom}(d_{H})$ is Fréchet differentiable, bounded and with differential $df$ bounded in the standard Finsler metric on $C_{x_{0}}M$.

Define $d_{H} : \text{Dom}(d_{H}) \subset L^{2}(C_{x_{0}}M; \mathbb{R}) \to L^{2}(\mathcal{H}^{*})$, to be the restriction of the Fréchet derivative to $\mathcal{H}$. Denote by $d = d_{C_{x_{0}}M}$ the closure of $d_{H}$ and by $\mathbb{D}^{2,1}(C_{x_{0}}M; \mathbb{R})$ the domain of $d$ with graph norm.

Consider the classical Wiener space $\Omega = C_{0}([0, T]; \mathbb{R}^{m})$ with Wiener measure $\mathbb{P}$, and a stochastic differential equation:

$$dx_{t} = X(x_{t}) \circ dB_{t} + A(x_{t}) \, dt \tag{1}$$

using canonical Brownian motion $[B_{t}(\omega) : 0 \leq t \leq T, \omega \in \Omega]$, where $X : \mathbb{R}^{m} \times M \to TM$ is a $C^{2}$ surjective bundle map and $A$ a smooth vector field. Assume it is a Brownian motion on $M$ and let $\mathcal{F}^{x_{0}}$ be its filtration. The solution starting from $x_{0}$ shall be denoted by $x_{t}(\omega)$, defined for almost all $\omega \in \Omega$. Denote by $Y(x) : T_{x}M \to \mathbb{R}^{m}$ the adjoint of $X(x)$.

It is shown in [7] that such an $s$ determines a metric connection $\tilde{\nabla}$ on $M$, the LJW connection, and that this is the Levi-Civita connection if and only if $X(x)(dY(v)) = 0$ all $v \in T_{x}M, x \in M$. This holds if (1) is the gradient $sd$ determined by an isometric immersion into $\mathbb{R}^{m}$.

There is the Itô map $\mathcal{I} : \Omega \to C_{x_{0}}M$,

$$\mathcal{I}_{t}(\omega) = x_{t}(\omega), \quad 0 \leq t \leq T, \tag{2}$$

which is measure preserving. Furthermore $\mathcal{I}_{t}$ is differentiable at $\omega$ in the direction of the Cameron–Martin space $H = L^{2,1}_{0}(\mathbb{R}^{m})$ in the sense of Malliavin calculus, giving a derivative of $\mathcal{I}$ which we write as $T_{\omega}\mathcal{I} : H \to T_{\omega}(C_{x_{0}}M; \mathbb{R})$. It also gives $\mathcal{I}^{*} : L^{2}(C_{x_{0}}M; \mathbb{R}) \to L^{2}(\Omega; \mathbb{R})$ by $\mathcal{I}^{*}(f) = f \circ \mathcal{I}$.

On $\Omega$ we also have the closed operator $d = d_{H} \circ L^{2}(\mathcal{H}^{*})$ with associated space $\mathbb{D}^{2,1}(\mathcal{H}^{*})$. Elements of $\mathbb{D}^{2,1}(C_{x_{0}}M; \mathbb{R})$ are characterised by a weak form of $H$-Gateaux differentiability in [14] and so $\mathbb{D}^{2,1}(C_{x_{0}}M; \mathbb{R})$ is independent of the choice of $\text{Dom}(d_{H})$ provided it satisfies the analogue of (i) and (ii). For $C_{x_{0}}M$ this independence has not been clear and a particular consequence of the results announced here is that it does hold.

Here we discuss only the case of Brownian motion measure, and Levi-Civita connections for brevity, but the proofs extend easily to the case of non-degenerate diffusions with constant rank symbols, and are given in detail in this context in [9]. For a discussion of intertwining properties of the stochastic development map see [4] (but the intertwining there is different from that discussed here) and [11]. Denote by $\mathcal{H}$ the ‘vector bundle’ with fibres $\mathcal{H}_{\sigma}$, by $L^{2}_{\mathcal{H}}$ the space of $L^{2}$ sections of $\mathcal{H}$, and by $\mathcal{H}^{*}$ the space of $L^{2}$ sections of the dual of $\mathcal{H}$.

This work draws on earlier work with S. Aida and Y. LeJan and was especially stimulated by our contacts with them, S. Fang and Z.-M. Ma. We are also grateful for comments by S. Fang on a preliminary version.

2. Main results

**Theorem 2.1.** Assume the LJW connection of (1) is the Levi-Civita connection. A real valued $L^{2} \text{ function } f \text{ on } C_{x_{0}}M \text{ belongs to } \text{Dom}(d_{C_{x_{0}}M}) \text{ if and only if } f \circ \mathcal{I} \in \text{Dom}(d_{\Omega})$. Consequently $\mathcal{I}^{*}$ gives a topological linear isomorphism of $\mathbb{D}^{2,1}(C_{x_{0}}M; \mathbb{R})$ with the closed subspace of $\mathbb{D}^{2,1}(\mathcal{H}^{*}) \text{ consisting of } \mathcal{F}^{x_{0}} \text{ measurable functions. Moreover } \mathbb{D}^{2,1}(\mathcal{H}^{*}) \text{ is mapped to itself by } \mathbb{E}[\cdot | \mathcal{F}^{x_{0}}]$. 
Idea of the proof. From [10], with a more general and corrected proof in [9], we know \( \mathcal{I}^* \) restricted to \( \mathbb{D}^{2,1}(C_{00} M; \mathbb{R}) \) has closed range in \( \mathbb{D}^{2,1}(\Omega; \mathbb{R}) \) and \( \mathcal{I}^* d_{C_{00} M} \subset d_{\Omega} \mathcal{I}^* \). We can therefore prove the result by showing that if \( \mathcal{I}^*(f) \in \mathbb{D}^{2,1}(\Omega; \mathbb{R}) \), the domain of \( d \) on \( \Omega \), then there exists a sequence \( \{f_n\}_{n=1}^\infty \) in \( \mathbb{D}^{2,1}(C_{00} M; \mathbb{R}) \) such that \( \mathcal{I}^*(f_n) \to \mathcal{I}^*(f) \) in \( \mathbb{D}^{2,1}(\Omega; \mathbb{R}) \). To do this we shall use the characterisation of \( \mathbb{D}^{2,1}(\Omega; \mathbb{R}) \) in terms of the chaos expansion and write \( \mathcal{I}^* \equiv G \) and to higher derivatives. Here we can only state some sample results. Details are in [9]. If \( \mathcal{I}^*(f_n) = 0 \) we have \( \mathcal{I}^*(f_n) = \sum_{k=1}^n J_k(\alpha_k) \) for \( J_k(\alpha_k) \) the iterated integral

\[
J_k(\alpha_k) := k! \int_0^t \int_0^{t_2} \cdots \int_0^{t_k} \langle \alpha_k(t_1, \ldots, t_k), K^\perp(x(t_1)) dB_{t_1} \otimes \cdots \otimes K^\perp(x(t_k)) B_{t_k} \rangle_{\mathbb{R}^m},
\]

where \( K^\perp(x) : \mathbb{R}^m \to \{\ker X(x)\}^\perp \) is the orthogonal projection \( Y(x)X(x) \), cf. [7, 2]. Using the fact that \( \mathbb{E}[|d(K^\perp \circ I_t)|^2] \) is in \( L^\infty(\Omega; \mathbb{R}) \) uniformly in \( t \), as in [2] we obtain the estimate \( \sum_k \|d(J_k(\alpha_k))\|_{L^2(\Omega, H^t)}^2 \leq \text{const.} \sum_k k! \|\alpha_k\|^2 \) which is finite, by Proposition 1.2.1 of Nualart [12], if \( \mathcal{I}^* f \) belongs to \( \mathbb{D}^{2,1}(\Omega; \mathbb{R}) \). This implies \( \mathcal{I}^*(f_n) \to \mathcal{I}^*(f) \) in \( \mathbb{D}^{2,1}(\Omega; \mathbb{R}) \).

On the other hand take \( \tilde{B}_t = \int_0^t \tilde{X}(x_s) d\tilde{B}_s \), the anti-development of \( x \) to see

\[
f_n(\sigma) = \sum_{k=1}^n k! \int_0^T \int_0^{t_2} \cdots \int_0^{t_k} \langle \alpha_k(t_1, \ldots, t_k), Y(x(t_1)) dB_{t_1} \otimes \cdots \otimes Y(x(t_k)) dB_{t_k} \rangle_{\mathbb{R}^m}.
\]

The fact that \( f_n \) is in \( \text{Dom}(d_{C_{00} M}) \) is essentially standard, e.g., see Cruzéirio–Malliavin [4] or the Appendix in Aida [1]. For a gradient stochastic differential equation (1) determined by an isometric \( J \) on smooth cylindrical functions \( \mathcal{I} \), Theorem 2.4. cf. [6, 13].

Corollary 2.2. \( \text{Dom}(d_{C_{00} M}) \) is independent of the choice of \( \text{Dom}(d_M) \) provided that it satisfies (i), (ii).

Corollary 2.3. (a) \( \mathcal{I}^* d_{C_{00} M} = d_{\Omega} \mathcal{I}^* \).

(b) There is equality of the following two forms: \( \int_{C_{00} M} |d_{C_{00} M} f|^2 d\mu_{x_0} = \int_{\Omega} |\mathbb{E}[d_{\Omega} \mathcal{I}^*(f)]|^2 \mu_{x_0} \) and there is a constant \( c \) with \( \int_{C_{00} M} |d_{C_{00} M} f|^2 d\mu_{x_0} \leq c \int_{\Omega} |d_{\Omega} \mathcal{I}^* f|^2 d\mu_{x_0} \leq c \int_{C_{00} M} |d_{C_{00} M} f|^2 d\mu_{x_0} \) for \( f \in \mathbb{D}^{2,1}(C_{00} M; \mathbb{R}) \), cf. [6, 13].

Using the characterisation of \( \text{Dom}(d_{\text{div}}) \) for \( \Omega \) in [12] (Proposition 1.3.1), Corollary 2.2 can be strengthened to:

Theorem 2.4. There is a unique closed operator \( d \) from \( L^2(C_{00} M; \mathbb{R}) \) to \( L^2(\Gamma H^t) \) such that \( d \) agrees with \( d_H \) on smooth cylindrical functions; (ii) \( \text{Dom}(d^t) \) contains all smooth cylindrical one forms.

2.1. Higher derivatives and covariant derivatives

The main result extends to covariant differentiation using the damped Markovian connection introduced in [3], and to higher derivatives. Here we can only state some sample results. Details are in [9]. If \( G \) is a separable Hilbert space we define \( d^G : \text{Dom}(d^G) \subset L^2(C_{00} M; G) \to L^2(\Gamma(\mathcal{H}^t \otimes G)) \) to be the closure of the derivative naturally defined with domain the linear span of \( \{F : C_{00} M \to \mathbb{R} | F(\sigma) = f(\sigma), g \in G \} \). Then the canonical isometry of \( L^2(C_{00} M; \mathbb{R}) \otimes G \) with \( L^2(C_{00} M; G) \) maps \( \text{Dom}(d^t) \otimes G \) onto \( \text{Dom}(d^G) \) so that Theorem 2.1 clearly holds for \( G \)-valued functions.

By an \( \mathcal{H}-1 \)-form we mean a section of \( \mathcal{H}^t \). Define \( \mathcal{W} : \text{Dom}(\mathcal{W}) \subset L^2(\mathcal{H}^t) \to L^2(\Gamma(\mathcal{H} \otimes \mathcal{H}^t)) \) by

\[
\mathcal{W} \phi(u) = \int \left( \phi \left( W \int W_x^{-1} X(e v_x) (-) dx \right) \right)(v) \left( Y(\sigma) \left( \frac{d}{d\sigma} u \right) \right) , \quad u, v \in \mathcal{H}_{x_0}.
\]
Dom(\(\mathbb{W}\)) = \(\{ \phi : \phi \left( \int_{0}^{T} W_{s}^{-1} X(\sigma(s)) \left( \frac{d}{ds} - \right) ds \right) \text{ is in } \text{Dom}(d) \} \).

With this domain \(\mathbb{W}\) is a closed operator, and is independent of the choice of the sde (1) provided it induces the Levi-Civita connection. From Theorem 2.1 and results in [8] on the conditional expectation of the \(H\)-derivative of an Itô map we have

**Corollary 2.5.** If \(\phi \in L^{2} \Gamma H^{*}\) then \(\phi \in \text{Dom}(\mathbb{W})\) if and only if \(I^{*}(\phi)\) has \(\mathbb{E}[I^{*}(\phi) | \mathcal{F}^{\omega}])\) in \(\text{Dom}(d)\).

Here the pull back \(I^{*}(\phi) = \phi \circ \mathcal{I}\) is defined as a limit in \(L^{2}\) of \((\phi_{n} \circ \mathcal{I})\) where the \(\phi_{n}\) are cylindrical and converge to \(\phi, [9,10]\). It can be treated as a stochastic integral.

We can extend the definition of \(\mathbb{W}\) to other \(H\)-tensors and define Sobolev spaces \(\mathbb{D}^{k} := (C_{\omega}, M; G)\) for \(k = 2, 3, \ldots\) in the usual way. These depend only on the Riemannian structure of \(M\).

**Definition 2.1.** Let \(\mathbb{D}^{2,1}_{\mathcal{F}^{\omega}}(\Omega; G)\) be the subset of \(\mathbb{D}^{1}(\Omega; G)\) whose elements are \(\mathcal{F}^{\omega}\)-measurable. Inductively \(\mathbb{D}^{2,1}_{\mathcal{F}^{\omega}}(\Omega; G)\) consists of \(F\) such that (a) \(F \in \mathbb{D}^{2,1}_{\mathcal{F}^{\omega}}(\Omega; G)\) and (b) \(\mathbb{E}[d_{\mathcal{F}} F | \mathcal{F}^{\omega}] : \Omega \to H^{*} \otimes G\) is in \(\mathbb{D}^{2,1}(\Omega; H^{*} \otimes G)\), furnished with the norm \(\|F\|_{\mathcal{F}^{\omega}, 2, k} := \left( \sum_{j=0}^{k} \| (d_{\mathcal{F}} \circ \mathbb{E}[ - | \mathcal{F}^{\omega}] )_{j} F \|_{L^{2}(\Omega; H^{*} \otimes G)}^{2} \right)^{1/2}\).

**Corollary 2.6.** An element of \(f \in L^{2}(C_{\omega}, M; \mathbb{R})\) is in \(\mathbb{D}^{2,1}(C_{\omega}, M; \mathbb{R})\) if and only if \(f \circ \mathcal{I}\) is in the domain of the \(k\)th iterate of the operator \(d_{\mathcal{F}} \circ \mathbb{E}[ - | \mathcal{F}^{\omega}]\). Consequently \(I^{*}\) restricts to give a linear isomorphism from \(\mathbb{D}^{2,1}(C_{\omega}, M; \mathbb{R})\) onto \(\mathbb{D}^{2,1}_{\mathcal{F}^{\omega}}(\Omega; \mathbb{R})\).

**References**


