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C. R. Acad. Sci. Paris, Ser. I 337 (2003) 805–808
COMPTES RENDUS MATHEMATIQUE

Numerical Analysis

Stability of discrete liftings

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Received 3 June 2003; accepted after revision 7 October 2003

Presented by Philippe G. Ciarlet

Abstract

In this short Note we prove the equivalence between having a discrete lifting of Dirichlet boundary conditions for (abstract) finite element spaces and having a Scott–Zhang type operator in the space, i.e., a stable projection preserving homogeneous boundary conditions. Both results are equivalent to the possibility of obtaining a Céa estimate where approximation of the boundary conditions is separated from the approximation capabilities of the space. ***To cite this article: V. Domínguez, F.-J. Sayas, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Stabilité des relèvements discrets. Dans cette courte Note nous démontrons l'équivalence entre l'existence d'un relèvement discret des conditions aux limites de Dirichlet pour un espace (abstrait) d'éléments finis et l'existence d'un opérateur de Scott–Zhang sur l'espace, c'est-à-dire, d'une projection stable qui préserve les conditions aux limites homogènes. Ces deux résultats sont équivalents à la possibilité d'obtenir une estimation de Céa, où l'approximation des conditions aux limites est séparée des propriétés d'approximation de l'espace. ***Pour citer cet article : V. Domínguez, F.-J. Sayas, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

1. Statement of the problem

Let V and M be Hilbert spaces, $\gamma : V \to M$ be a bounded surjective linear operator (the abstract trace) and $V^0 = \ker\gamma$. Let $a : V \times V \to \mathbb{C}$ be a bounded sesquilinear form. We consider the following problem: given $\eta \in M$, find the solution to

$$\begin{cases} u \in V, \quad \gamma u = \eta, \\ a(u, v) = 0, \quad \forall v \in V^0. \end{cases} \tag{1}$$

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2003.10.025

Remark 1. To keep notations as simple as possible, in the following we will be using the same symbol for norms, $\|\cdot\|$, and inner products, $(\cdot,\cdot)$, of M and V. It is the notation for the elements (Greek letters for elements of M and Latin for those of V) that will make the context clear.

To ensure well-posedness of (1) we assume the following:

Hypothesis I. *The operator $A_0 : V^0 \to V^0$ defined by the relation $(A_0 u, v) = a(u,v)$, for all $u, v \in V^0$, is invertible.*

If this hypothesis holds, then (1) has a unique solution. We define $R : M \to V$ to the operator such that $R\eta := u$, the solution of (1). It is clear that R is bounded and is a right-inverse for γ. We will call it a lifting. In particular, if we take the inner product of V as sesquilinear form, the associated lifting is just the pseudoinverse of γ, which we denote γ^+ (see [2]).

Remark 2. The standard example for this abstract setting consists of taking $V = H^1(\Omega)$, $M = H^{1/2}(\Gamma)$, γ the trace operator (and thus $V_0 = H_0^1(\Omega)$) and $a(\cdot,\cdot)$ being the sesquilinear form associated to an elliptic operator, for instance, $a(u,v) = \int_\Omega \nabla u \cdot \nabla \bar{v}$.

Let now $V_h \subset V$ be a family of finite dimensional subspaces of V and consider the spaces $V_h^0 := V_h \cap V^0$ and $M_h := \gamma V_h$. We then consider the discretized version of (1): given $\eta_h \in M_h$ (in practice one takes $\eta_h \approx \eta$ in some way), solve:

$$\begin{cases} u_h \in V_h, \quad \gamma u_h = \eta_h, \\ a(u_h, v_h) = 0, \quad \forall v_h \in V_h^0. \end{cases} \tag{2}$$

The discretized version of Hypothesis I is:

Hypothesis II. *There exists $\alpha > 0$ such that*

$$\sup_{0 \neq u_h \in V_h^0} \frac{|a(u_h, v_h)|}{\|u_h\|} \geqslant \alpha \|v_h\|, \quad \forall v_h \in V_h^0. \tag{3}$$

If this hypothesis holds, it is very simple to prove that (2) has a unique solution and that there exists $C_0 > 0$ such that

$$\|u - u_h\| \leqslant C_0 \inf\bigl\{\|u - v_h\| \mid v_h \in V_h,\ \gamma v_h = \eta_h\bigr\}. \tag{4}$$

The operator mapping η_h to u_h will be denoted $R_h : M_h \to V_h$. Again, in case the sesquilinear form is the inner product, Hypothesis II trivially holds and the operator, denoted by γ_h^+, is just the pseudoinverse of $\gamma_h := \gamma|_{V_h} : V_h \to M_h$.

Remark 3. There are two simple cases where both hypotheses hold.

(a) The sesquilinear form is V_0-elliptic, i.e., there exists $\alpha > 0$ such that $\operatorname{Re} a(u,u) \geqslant \alpha \|u\|^2$ for all $u \in V_0$.
(b) There exists a Hilbert space H, such that $V \subset H$ with dense compact inclusion, $a(\cdot,\cdot)$ satisfies a Garding inequality (here $\alpha, \kappa > 0$):

$$\operatorname{Re} a(u,u) \geqslant \alpha \|u\|^2 - \kappa \|u\|_H^2, \quad \forall u \in V^0 \tag{5}$$

and the homogeneous version of (1) does not admit but the trivial solution, then Hypothesis I is satisfied and (3) holds for h small enough provided that for all $u \in V^0$, $\inf_{v_h \in V_h^0} \|u - v_h\| \to 0$.

In the remainder of the paper, we will assume that Hypotheses I and II hold.

2. Main results

Theorem 2.1. *The following statements are equivalent*:

(1) R_h *is uniformly bounded*;
(2) γ_h^+ *is uniformly bounded*;
(3) *there exists* $L_h : M_h \to V_h$ *linear uniformly bounded satisfying* $\gamma L_h \eta_h = \eta_h$ *for all* $\eta_h \in M_h$.

Proof. Notice that $\|\gamma_h^+ \eta_h\| \leqslant \|v_h\|$ for any $v_h \in V_h$ such that $\gamma v_h = \eta_h$. Then we just have to prove that uniform boundedness of R_h is implied by that of γ_h^+. This last is, however equivalent to the following discrete uniform Babuška–Brezzi type condition (see [1]): there exists $\beta > 0$ such that

$$\sup_{0 \neq v_h \in V_h} \frac{|(\gamma v_h, \mu_h)|}{\|v_h\|} \geqslant \beta \|\mu_h\|, \quad \forall \mu_h \in M_h. \tag{6}$$

Since $(\gamma v_h, \mu_h) = 0$ for all $\mu_h \in M_h$ implies that $\gamma v_h = 0$, then $u_h := R_h g_h$ solves:

$$\begin{cases} u_h \in V_h, \lambda_h \in M_h, \\ a(u_h, v_h) + (\lambda_h, \gamma v_h) = 0, \quad \forall v_h \in V_h, \\ (\gamma u_h, \mu_h) = (\eta_h, \mu_h), \quad \forall \mu_h \in M_h. \end{cases}$$

Then (3) and (6) show that $\|u_h\| + \|\lambda_h\| \leqslant C\|\eta_h\|$, with a constant C depending on α and β. □

Hypothesis III. *For all* h, *there exists an operator* $\Pi_h : V \to V_h$, *such that, if it is uniformly bounded, it is a projection onto* V_h *and if* $u \in V^0$, *then* $\Pi_h u \in V_h^0$ (*it respects the boundary condition* $\gamma u = 0$).

Two of these operators have been studied in [4] and [3], for particular choices of finite element spaces.

Theorem 2.2. *If Hypothesis* III *holds, then* R_h *is uniformly bounded.*

Proof. Let $\eta_h \in M_h$ and consider $u := R\eta_h$ (i.e., problem (1) with $\eta = \eta_h$) and $u_h := R_h \eta_h$, the solution of (2). Since $u - u_h \in V^0$, then $\Pi_h u - u_h = \Pi_h(u - u_h) \in V_h^0$ and we can take $\Pi_h u$ in (4):

$$\|u - u_h\| \leqslant C_0 \|u - \Pi_h u\| \leqslant C_0 \big(1 + \|\Pi_h\|\big) \inf_{v_h \in V_h} \|u - v_h\| \leqslant C_1 \|u\|.$$

This easily gives the result. □

Remark 4. Notice that existence of Π_h satisfying Hypothesis III allows to prove a variant of the Céa estimate (4),

$$\|u - u_h\| \leqslant C_2 \inf_{v_h \in V_h} \|u - v_h\| + C_3 \|\eta - \eta_h\|. \tag{7}$$

This allows for a simple approach to the analysis of the approximation of (1) by (2), even with non-homogeneous right-hand side.

Theorem 2.3. *If* γ_h^+ *is uniformly bounded, then there exists* Π_h *in the conditions of Hypothesis* III.

Proof. Let $P_h^0 : V \to V_h^0$ and $T_h : M \to M_h$ be the orthogonal projections onto V_h^0 and M_h respectively. Then

$$\Pi_h u := P_h^0 u + \gamma_h^+ T_h \gamma u.$$

It is clear that Π_h is uniformly bounded and that if $u \in V^0$ (that is, $\gamma u = 0$) then $\Pi_h u = P_h^0 u \in V_h^0$. If $u_h \in V_h$, then $T_h \gamma u_h = \gamma u_h$ and thus

$$\Pi_h u_h = P_h^0 u_h + \gamma_h^+ \gamma u_h = v_h^0 + v_h^1,$$

where

$$\begin{cases} v_h^0 \in V_h, \quad \gamma v_h^0 = 0, \\ (v_h^0, v_h) = (u_h, v_h), \quad \forall v_h \in V_h^0 \end{cases} \quad \text{and} \quad \begin{cases} v_h^1 \in V_h, \quad \gamma v_h^1 = \gamma u_h, \\ (v_h^1, v_h) = 0, \quad \forall v_h \in V_h^0. \end{cases}$$

Therefore $\Pi_h u_h$ satisfies:

$$\begin{cases} \Pi_h u_h \in V_h, \quad \gamma \Pi_h u_h = \gamma u_h, \\ (\Pi_h u_h, v_h) = (u_h, v_h), \quad \forall v_h \in V_h^0, \end{cases}$$

and by uniqueness of solution $\Pi_h u_h = u_h$. □

Remark 5. The theory of mixed methods gives also some additional insight into this matter. Assume there exists an operator $\Pi_h : V \to V_h$ satisfying the requirements of Fortin's lemma: uniform boundedness and compatibility, $(\gamma \Pi_h u, \mu_h) = (\gamma u, \mu_h), \ \forall \mu_h \in M_h$. Then, if this operator is a projection onto V_h, it also satisfies Hypothesis III.

3. Two simple consequences

The first by-product of these results is a simplified version of the Céa estimate, provided that the choice $\eta_h \approx \eta$ is stable. Obviously, if the sequence V_h satisfies an approximation property in V, then this implies convergence of the solutions of (2) to that of (1).

Corollary 3.1. *Assume that $N_h : M \to M_h$ is a uniformly bounded projection onto M_h. If R_h is uniformly bounded and we take $\eta_h = N_h \eta$ in* (2)*, then $\|u - u_h\| \leqslant C_4 \inf_{v_h \in V_h} \|u - v_h\|$.*

Proof. Let w_h the best approximation of u in V_h, i.e., $\|u - w_h\| = \inf_{v_h \in V_h} \|u - v_h\|$. Then

$$\|\eta - \eta_h\| \leqslant \big(1 + \|N_h\|\big) \inf_{\rho_h \in M_h} \|\eta - \rho_h\| \leqslant \big(1 + \|N_h\|\big)\|\eta - \gamma w_h\| \leqslant \big(1 + \|N_h\|\big)\|\gamma\| \|u - w_h\|.$$

The result then follows by (7). □

The associated Dirichlet-to-Neumann operator in this abstract setting is the mapping $M \to M'$ given by:

$$\eta \mapsto a(R\eta, R\cdot) = a(R\eta, \gamma^+ \cdot) : M \to \mathbb{C}.$$

The final result proves uniform boundedness of the discretization of this operator between abstract Cauchy data. Its proof is straightforward.

Corollary 3.2. *If R_h is uniformly bounded, then the discrete operator $M_h \to M_h'$ given by $\eta_h \mapsto a(R_h \eta_h, R_h \cdot) : M_h \to \mathbb{C}$ is uniformly bounded.*

Acknowledgements

The authors are partially supported by MCYT/FEDER Projects BFM2001-2521 and MAT2002-04153 and by Gobierno de Navarra Ref. 134/2002.

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