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C. R. Acad. Sci. Paris, Ser. I 338 (2004) 71–76
COMPTES RENDUS MATHEMATIQUE

Differential Geometry/Mathematical Problems in Mechanics

On the fundamental theorem of surface theory under weak regularity assumptions

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Received 22 September 2003; accepted 4 October 2003

Presented by Philippe G. Ciarlet

Abstract

We consider a symmetric, positive definite matrix field of order two and a symmetric matrix field of order two that together satisfy the Gauss and Codazzi–Mainardi equations in a connected and simply connected open subset of $\mathbb{R}^2$. If these fields are of class C^2 and C^1 respectively, the fundamental theorem of surface theory asserts that there exists a surface immersed in the three-dimensional Euclidean space with the given matrix fields as its first and second fundamental forms. The purpose of this Note is to prove that this theorem still holds true under the weaker regularity assumptions that these fields are of class $W^{1,\infty}_{\mathrm{loc}}$ and $L^{\infty}_{\mathrm{loc}}$ respectively, the Gauss and Codazzi–Mainardi equations being then understood in a distributional sense. ***To cite this article: S. Mardare, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Sur le théorème fondamental de la théorie des surfaces sous des hypothèses faibles de régularité. On considère un champ de matrices symétriques définies positives d'ordre deux et un champ de matrices symétriques d'ordre deux qui satisfont ensemble les équations de Gauss et de Codazzi–Mainardi dans un ouvert connexe et simplement connexe de $\mathbb{R}^2$. Si ces champs sont respectivement de classe C^2 et C^1, alors le théorème fondamental de la théorie des surfaces affirme qu'il existe une surface plongée dans l'espace Euclidean tridimensionnel dont ces champs sont les première et deuxième formes fondamentales. L'objet de cette Note est d'établir que ce théorème reste vrai sous les hypothèses de régularités affaiblies selon lesquelles ces champs sont respectivement de classe $W^{1,\infty}_{\mathrm{loc}}$ et $L^{\infty}_{\mathrm{loc}}$, les équations the Gauss et de Codazzi–Mainardi étant alors satisfaites aux sens des distributions. ***Pour citer cet article : S. Mardare, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

On considère un champ $(a_{\alpha\beta})$ de matrices symétriques définies positives d'ordre deux et un champ $(b_{\alpha\beta})$ de matrices symétriques d'ordre deux définis dans un ouvert connexe et simplement connexe ω de $\mathbb{R}^2$. On suppose que les fonctions $a_{\alpha\beta}$ et $b_{\alpha\beta}$ sont respectivement de classe C^2 et C^1 dans ω et qu'elles satisfont ensemble les équations de Gauss et de Codazzi–Mainardi, à savoir

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2003.10.027

$$\partial_\gamma \Gamma^\tau_{\alpha\beta} - \partial_\beta \Gamma^\tau_{\alpha\gamma} + \Gamma^\sigma_{\alpha\beta}\Gamma^\tau_{\sigma\gamma} - \Gamma^\sigma_{\alpha\gamma}\Gamma^\tau_{\sigma\beta} = b_{\alpha\beta} b^\tau_\gamma - b_{\alpha\gamma} b^\tau_\beta,$$
$$\partial_\gamma b_{\alpha\beta} - \partial_\beta b_{\alpha\gamma} + \Gamma^\sigma_{\alpha\beta} b_{\sigma\gamma} - \Gamma^\sigma_{\alpha\gamma} b_{\sigma\beta} = 0.$$

Les symboles de Christoffel $\Gamma^\tau_{\alpha\beta}$ associés à la métrique $(a_{\alpha\beta})$, ainsi que d'autre notations, sont définis dans la version anglaise.

Sous ces hypothèses, le théorème fondamental de la théorie des surfaces (voir, e.g., [1,3,4]) affirme qu'il existe une application $\boldsymbol{\theta} : \omega \to \mathbb{R}^3$ de classe C^3 telle que les première et deuxième formes fondamentales de la surface $S = \boldsymbol{\theta}(\omega)$ sont respectivement données par les champs des matrices $(a_{\alpha\beta})$ et $(b_{\alpha\beta})$, i.e.,

$$\partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} = a_{\alpha\beta} \quad \text{et} \quad \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} = b_{\alpha\beta} \quad \text{dans } \omega.$$

L'objet de cette Note est d'établir que le théorème fondamental de la théorie des surfaces reste vrai sous des hypothèses de régularité plus faibles. Le résultat principal (voir Théorème 2.1 dans la version anglaise) s'énonce ainsi : si les champs de matrices $(a_{\alpha\beta})$ et $(b_{\alpha\beta})$ sont respectivement de classe $W^{1,\infty}_{\mathrm{loc}}$ et L^∞_{loc} dans ω et satisfont ensemble les équations de Gauss et de Codazzi–Mainardi au sens des distributions, alors il existe une application $\boldsymbol{\theta} : \omega \to \mathbb{R}^3$ de classe $W^{2,\infty}_{\mathrm{loc}}$ telle que les première et deuxième formes fondamentales de la surface $S = \boldsymbol{\theta}(\omega)$ sont respectivement données par les champs de matrices $(a_{\alpha\beta})$ et $(b_{\alpha\beta})$. De plus, si le diamètre géodésique de l'ouvert ω est fini, $a_{\alpha\beta} \in W^{1,\infty}(\omega)$, $b_{\alpha\beta} \in L^\infty(\omega)$, et $(a_{\alpha\beta})^{-1} \in L^\infty(\omega, \mathbb{M}^2)$, alors l'application $\boldsymbol{\theta}$ appartient à l'espace $W^{2,\infty}(\omega, \mathbb{R}^3)$.

La démonstration complète de ce théorème, esquissée dans la version anglaise, se trouve dans [6].

1. Preliminaries

All functions and fields appearing in this paper are real valued and the summation convention with respect to repeated indices and exponents is used.

For any integer $d \geqslant 2$, the d-dimensional Euclidean space will be identified with $\mathbb{R}^d$. Let $\boldsymbol{u} \cdot \boldsymbol{v}$ denote the Euclidean inner product of $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d$ and let $|\boldsymbol{v}|$ denote the Euclidean norm of $\boldsymbol{v} \in \mathbb{R}^d$.

Let $\mathbb{M}^{q,l}$ denote the set of all matrices with q rows and l columns and let $\mathbb{M}^d := \mathbb{M}^{d,d}$. The notations $\mathbb{S}^d$ and $\mathbb{S}^d_>$ respectively designate the set of all symmetric matrices, and of all positive definite symmetric matrices, of order d. The notation (a_{ij}) (or (a^i_j)) designates the matrix whose entries are the elements a_{ij} (or a^i_j), where the first (or upper) index is the row index and the second (or lower) index is the column index.

Let $x = (x_1, x_2, \ldots, x_d)$ denote a generic point in $\mathbb{R}^d$ and let $\partial_i := \partial/\partial x_i$. The open ball centered at $x \in \mathbb{R}^d$ of radius $R > 0$ is denoted $B(x, R)$ and $|B(x, R)|$ denotes the $\mathbb{R}^d$-Lebesgue measure of $B(x, R)$. The distance between two subsets A and B of $\mathbb{R}^d$ is defined by

$$\operatorname{dist}(A, B) = \inf_{x \in A,\, y \in B} |x - y|.$$

The geodesic diameter of an open subset Ω of $\mathbb{R}^d$ is the number $D_\Omega \in [0, \infty]$ defined by

$$D_\Omega := \sup_{x, y \in \Omega} \inf_{\boldsymbol{\gamma}} L(\boldsymbol{\gamma}),$$

where $\boldsymbol{\gamma} \in C^0([0, 1]; \Omega)$ is any path joining x to y and $L(\boldsymbol{\gamma})$ designates the length of the path $\boldsymbol{\gamma}$.

Let $\mathcal{D}'(\Omega)$ denote the space of distributions defined over Ω, let $W^{m,p}(\Omega; \mathbb{M}^{q,l})$ denote the usual Sobolev space, and let

$$W^{m,p}_{\mathrm{loc}}(\Omega; \mathbb{M}^{q,l}) := \{\boldsymbol{v} \in \mathcal{D}'(\Omega; \mathbb{M}^{q,l});\ \boldsymbol{v} \in W^{m,p}(U; \mathbb{M}^{q,l}) \text{ for all open set } U \Subset \Omega\},$$

where the notation $U \Subset \Omega$ means that the closure of U in $\mathbb{R}^d$ is a compact subset of Ω. For real-valued function spaces we shall use the notation $W^{m,p}(\Omega)$ instead of $W^{m,p}(\Omega, \mathbb{R})$, $\mathcal{D}'(\Omega)$ instead of $\mathcal{D}'(\Omega, \mathbb{R})$, etc.

In what follows, we make the following convention for classes of functions with respect to the equality almost everywhere: if $\dot{f} \in L^\infty_{\rm loc}(\Omega)$, we will always use the representative f of $\dot{f}$ given by

$$f(x) := \liminf_{\varepsilon\to 0} \frac{1}{|B(x,\varepsilon)|} \int\limits_{B(x,\varepsilon)} \tilde{f}(y)\,{\rm d}y,$$

where $\tilde{f}$ is any representative of the class $\dot{f} \in L^\infty_{\rm loc}(\Omega)$ (this definition is clearly independent of the choice of the representative $\tilde{f}$). This choice of the representative insures that $\|\dot{f}\|_{L^\infty(U)} = \sup_{x\in U} |f(x)|$ for all open subset U of Ω. Also, for any $\dot{f} \in W^{1,\infty}_{\rm loc}(\Omega)$, we will choose the continuous representative f of $\dot{f}$. For simplicity, we will use the same notation for a class of functions and its representative chosen as before, the distinction between them being made according to the context.

The following result constitutes a key step towards establishing the main result of this Note, viz., Theorem 2.1.

Theorem 1.1. *Let Ω be a connected and simply connected open subset of $\mathbb{R}^d$ and let a point $x^0 \in \Omega$ and a matrix $Y^0 \in \mathbb{M}^{q,l}$ be fixed. Let the matrix fields $A_\alpha \in L^\infty_{\rm loc}(\Omega; \mathbb{M}^l)$ and $B_\alpha \in L^\infty_{\rm loc}(\Omega; \mathbb{M}^{q,l})$ be given such that*

$$\partial_\alpha A_\beta + A_\alpha A_\beta = \partial_\beta A_\alpha + A_\beta A_\alpha \quad \text{in } \mathcal{D}'\big(\Omega; \mathbb{M}^l\big) \quad \text{for all } \alpha, \beta \in \{1, 2, \ldots, d\},$$
$$\partial_\alpha B_\beta + B_\alpha A_\beta = \partial_\beta B_\alpha + B_\beta A_\alpha \quad \text{in } \mathcal{D}'\big(\Omega; \mathbb{M}^{q,l}\big) \quad \text{for all } \alpha, \beta \in \{1, 2, \ldots, d\}.$$

(i) *Then there exists a unique solution $Y \in W^{1,\infty}_{\rm loc}(\Omega; \mathbb{M}^{q,l})$ to the system*

$$\partial_\alpha Y = Y A_\alpha + B_\alpha \quad \text{for all } \alpha \in \{1, 2, \ldots, d\}, \tag{1}$$
$$Y\big(x^0\big) = Y^0. \tag{2}$$

(ii) *If in addition the geodesic diameter of the set Ω is finite and the matrix fields A_α and B_α respectively belong to $L^\infty(\Omega; \mathbb{M}^l)$ and $L^\infty(\Omega; \mathbb{M}^{q,l})$, then the solution Y to the above system belongs to $W^{1,\infty}(\Omega; \mathbb{M}^{q,l})$.*

Sketch of proof. The proof of part (i) is based on techniques similar to those used for establishing Theorem 2.1 of [5], but generalized to systems of the form (1) with coefficients A_α, B_α in $L^\infty_{\rm loc}$ over Ω (in [5], we considered systems of the form (1) with coefficients $A_\alpha \in L^\infty$ over Ω and $B_\alpha = 0$, the elements of the matrix field A_α being the Christoffel symbols associated with a Riemannian metric in Ω).

(ii) Let $c_1, c_2 \geqslant 0$ be two constants such that

$$\|A_\alpha\|_{L^\infty(\Omega;\mathbb{M}^l)} \leqslant c_1 \quad \text{and} \quad \|B_\alpha\|_{L^\infty(\Omega;\mathbb{M}^{q,l})} \leqslant c_2.$$

Let $x \in \Omega$ be a fixed, but otherwise arbitrary, point in Ω. Given $\varepsilon > 0$, the definition of the geodesic diameter D_Ω shows that there exists a path $\tilde{\boldsymbol{\gamma}}$ joining x^0 to x such that $L(\tilde{\boldsymbol{\gamma}}) \leqslant D_\Omega + \varepsilon$.

Let $r = \frac{1}{4}\,{\rm dist}(\tilde{\boldsymbol{\gamma}}([0,1]), \Omega^c) > 0$, where $\Omega^c = \mathbb{R}^d \setminus \Omega$. Since $\tilde{\boldsymbol{\gamma}}$ is uniformly continuous over [0, 1], there exists numbers $t_0, t_1, t_2, \ldots, t_N \in [0,1]$ such that $0 = t_0 < t_1 < t_2 < \cdots < t_N = 1$ and

$$\big|\tilde{\boldsymbol{\gamma}}(t_i) - \tilde{\boldsymbol{\gamma}}(t_{i-1})\big| < r \quad \text{for all } i \in \{1, 2, \ldots, N\}.$$

Let $\delta := \min(r, \frac{\varepsilon}{2N})$. Then there exist points $x^1, x^2, \ldots, x^N \in \Omega \cap B(\tilde{\boldsymbol{\gamma}}(t_i), \delta)$ such that

$$Y\big(\boldsymbol{\gamma}(t)\big) = Y\big(\boldsymbol{\gamma}(0)\big) + \int\limits_0^t \big\{(Y A_\alpha)\big(\boldsymbol{\gamma}(s)\big) + B_\alpha\big(\boldsymbol{\gamma}(s)\big)\big\}\gamma'_\alpha(s)\,{\rm d}s \quad \text{for all } t \in [0, N],$$

where $\boldsymbol{\gamma} = (\gamma_\alpha) : [0, N] \to \Omega$ maps the interval $[i-1, i]$ onto the segment $[x^{i-1}, x^i]$ for all $i \in \{1, 2, \ldots, N\}$. Consequently,

$$\left|Y\big(\boldsymbol{\gamma}(t)\big)\right| \leqslant \big(\left|Y^0\right| + c_2\sqrt{d}L(\boldsymbol{\gamma})\big) + c_1\sqrt{d}\int_0^t \left|\boldsymbol{\gamma}'(s)\right|\left|Y\big(\boldsymbol{\gamma}(s)\big)\right| \mathrm{d}s \quad \text{for all } t \in [0, N],$$

which next implies, by Gronwall inequality, that

$$\left|Y\big(x^N\big)\right| \leqslant \big(\left|Y^0\right| + c_2\sqrt{d}L(\boldsymbol{\gamma})\big)\, \mathrm{e}^{c_1\sqrt{d}L(\boldsymbol{\gamma})}.$$

But the length of the path $\boldsymbol{\gamma}$ satisfies $L(\boldsymbol{\gamma}) \leqslant L(\tilde{\boldsymbol{\gamma}}) + \varepsilon \leqslant D_\Omega + 2\varepsilon$ thanks to the triangular inequality. Letting then ε go to zero, we deduce from the above inequality (where the terms x^N and $L(\boldsymbol{\gamma})$ depend on ε) that

$$\left|Y(x)\right| \leqslant \big(\left|Y^0\right| + c_2\sqrt{d}D_\Omega\big)\, \mathrm{e}^{c_1\sqrt{d}D_\Omega},$$

since x^N goes to x when ε goes to zero. This implies that the matrix field Y belongs to the space $L^\infty(\Omega, \mathbb{M}^{q,l})$ and, since the derivatives of the field Y are given by $\partial_\alpha Y = YA_\alpha + B_\alpha$, the field Y also belongs to the space $W^{1,\infty}(\Omega, \mathbb{M}^{q,l})$. □

2. The fundamental theorem of surface theory revisited

Throughout this section, Greek indices vary in the set $\{1, 2\}$, Latin indices vary in the set $\{1, 2, 3\}$ and the summation convention with respect to repeted indices is used in conjunction with these rules.

Let ω be a connected and simply-connected open subset of $\mathbb{R}^2$ and let $y = (y_1, y_2)$ denote a generic point in ω. Let there be given two matrix fields $(a_{\alpha\beta}) \in W^{1,\infty}_{\mathrm{loc}}(\omega; \mathbb{S}^2_>)$ and $(b_{\alpha\beta}) \in L^\infty_{\mathrm{loc}}(\omega; \mathbb{S}^2)$ and define the Christoffel symbols associated with $(a_{\alpha\beta})$ by letting

$$\Gamma^\tau_{\alpha\beta} = \frac{1}{2}a^{\tau\sigma}(\partial_\alpha a_{\beta\sigma} + \partial_\beta a_{\sigma\alpha} - \partial_\sigma a_{\alpha\beta}),$$

where $(a^{\tau\sigma}(y))$ is the inverse of the matrix $(a_{\alpha\beta}(y))$ and $\partial_\alpha = \partial/\partial y_\alpha$. We recall that, according to the conventions made in Section 1, the field $(a_{\alpha\beta})$ is the continuous representative of the class, still denoted by, $(a_{\alpha\beta})$. Therefore, $(a_{\alpha\beta}(y)) \in \mathbb{S}^2_>$ for all $y \in \omega$. This implies that the inverse matrix $(a^{\tau\sigma})$ is continuous over ω and that the Christoffel symbols $\Gamma^\tau_{\alpha\beta}$ belong to the space $L^\infty_{\mathrm{loc}}(\omega)$.

Assume that the Gauss and Codazzi–Mainardi equations,

$$\partial_\gamma \Gamma^\tau_{\alpha\beta} - \partial_\beta \Gamma^\tau_{\alpha\gamma} + \Gamma^\sigma_{\alpha\beta}\Gamma^\tau_{\sigma\gamma} - \Gamma^\sigma_{\alpha\gamma}\Gamma^\tau_{\sigma\beta} = b_{\alpha\beta}b^\tau_\gamma - b_{\alpha\gamma}b^\tau_\beta,$$
$$\partial_\gamma b_{\alpha\beta} - \partial_\beta b_{\alpha\gamma} + \Gamma^\sigma_{\alpha\beta}b_{\sigma\gamma} - \Gamma^\sigma_{\alpha\gamma}b_{\sigma\beta} = 0,$$

are satisfied in $\mathcal{D}'(\omega)$ (these equations make sense in $\mathcal{D}'(\omega)$ since $\Gamma^\tau_{\alpha\beta}$, $b_{\alpha\beta}$ and $a^{\sigma\tau}$ belong to $L^\infty_{\mathrm{loc}}(\omega)$).

Our aim is to prove the existence of a surface immersed in the three-dimensional Euclidean space whose first and second fundamental forms are given by the matrix fields $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$, respectively. Our main result is the following:

Theorem 2.1. *Assume that ω is a connected and simply-connected open subset of $\mathbb{R}^2$ and that the matrix fields $(a_{\alpha\beta}) \in W^{1,\infty}_{\mathrm{loc}}(\omega; \mathbb{S}^2_>)$ and $(b_{\alpha\beta}) \in L^\infty_{\mathrm{loc}}(\omega; \mathbb{S}^2)$ satisfy the Gauss and Codazzi–Mainardi equations in $\mathcal{D}'(\omega)$. Then there exists a mapping $\boldsymbol{\theta} \in W^{2,\infty}_{\mathrm{loc}}(\omega, \mathbb{R}^3)$, unique up to proper isometries in $\mathbb{R}^3$, such that*

$$a_{\alpha\beta} = \partial_\alpha\boldsymbol{\theta} \cdot \partial_\beta\boldsymbol{\theta} \quad \text{and} \quad b_{\alpha\beta} = \partial_{\alpha\beta}\boldsymbol{\theta} \cdot \frac{\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}}{|\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}|} \quad \text{a.e. in } \omega.$$

If the geodesic diameter of the set ω is finite, $(a_{\alpha\beta}) \in W^{1,\infty}(\omega; \mathbb{S}^2_>)$, $(b_{\alpha\beta}) \in L^\infty(\omega; \mathbb{S}^2)$, and $(a_{\alpha\beta})^{-1} \in L^\infty(\omega, \mathbb{M}^2)$, then the mapping $\boldsymbol{\theta}$ belongs to $W^{2,\infty}(\omega, \mathbb{R}^3)$.

Sketch of proof. The proof is broken into six steps numbered (i) to (vi). Throughout the proof, we fix a point $y^0 \in \omega$, a vector $\boldsymbol{\theta}^0 \in \mathbb{R}^3$, and two vectors $\boldsymbol{a}_\alpha^0 \in \mathbb{R}^3$ such that $\boldsymbol{a}_\alpha^0 \cdot \boldsymbol{a}_\beta^0 = a_{\alpha\beta}(y^0)$. We also define a unit normal vector to $\boldsymbol{a}_\alpha^0$ by letting $\boldsymbol{a}_3^0 := (\boldsymbol{a}_1^0 \wedge \boldsymbol{a}_2^0)/|\boldsymbol{a}_1^0 \wedge \boldsymbol{a}_2^0|$. Finally, we define the matrix fields $\Gamma_\alpha : \omega \to \mathbb{M}^3$ by letting

$$\Gamma_\alpha := \begin{pmatrix} \Gamma_{\alpha 1}^1 & \Gamma_{\alpha 2}^1 & -b_\alpha^1 \\ \Gamma_{\alpha 1}^2 & \Gamma_{\alpha 2}^2 & -b_\alpha^2 \\ b_{\alpha 1} & b_{\alpha 2} & 0 \end{pmatrix}. \tag{3}$$

(i) *The Gauss and Codazzi–Mainardi equations are satisfied if and only if the following matrix equation is satisfied*:

$$\partial_\alpha \Gamma_\beta + \Gamma_\alpha \Gamma_\beta = \partial_\beta \Gamma_\alpha + \Gamma_\beta \Gamma_\alpha \quad \textit{in } \mathcal{D}'(\omega; \mathbb{M}^3),\ \alpha, \beta \in \{1, 2\}.$$

Note that this equation make sense since $\Gamma_\alpha \in L^\infty_{\text{loc}}(\omega; \mathbb{M}^3) \subset \mathcal{D}'(\omega; \mathbb{M}^3)$. It suffices to prove that the Gauss and Codazzi–Mainardi equations imply the relation

$$\partial_\alpha b_\beta^\tau + \Gamma_{\alpha\gamma}^\tau b_\beta^\gamma = \partial_\beta b_\alpha^\tau + \Gamma_{\beta\gamma}^\tau b_\alpha^\gamma.$$

From the definition of the Christoffel symbols, we deduce that $\partial_\alpha a_{\sigma\tau} = \Gamma_{\alpha\sigma}^\gamma a_{\tau\gamma} + \Gamma_{\alpha\tau}^\gamma a_{\sigma\gamma}$. Using this relation and the identity $b_\alpha^\tau a_{\tau\gamma} = b_{\alpha\gamma}$ in the Codazzi–Mainardi equations, we obtain that

$$a_{\sigma\tau} \partial_\alpha b_\beta^\tau + b_{\beta\gamma} \Gamma_{\alpha\sigma}^\gamma + b_\beta^\tau \Gamma_{\alpha\tau}^\gamma a_{\sigma\gamma} + \Gamma_{\beta\sigma}^\gamma b_{\alpha\gamma} = a_{\sigma\tau} \partial_\beta b_\alpha^\tau + b_{\alpha\gamma} \Gamma_{\beta\sigma}^\gamma + b_\alpha^\tau \Gamma_{\beta\tau}^\gamma a_{\sigma\gamma} + \Gamma_{\alpha\sigma}^\gamma b_{\beta\gamma}.$$

Multiplying the previous relation with $a^{\sigma\psi}$ gives the desired relation.

(ii) *There exists a solution $F \in W^{1,\infty}_{\text{loc}}(\omega, \mathbb{M}^3)$ to the system*

$$\partial_\alpha F = F \Gamma_\alpha \quad \textit{in } \mathcal{D}'(\omega; \mathbb{M}^3) \quad \textit{and} \quad F(y^0) = F^0, \tag{4}$$

where $F^0 \in \mathbb{M}^3$ is the matrix whose i-th column is $\boldsymbol{a}_i^0 \in \mathbb{R}^3$.

It suffices to apply Theorem 1.1 to the above system, the assumptions of this theorem being satisfied thanks to the previous step.

(iii) *Let $\boldsymbol{a}_i$ denote the i-th column of the matrix field F. Then there exists a solution $\boldsymbol{\theta} \in W^{2,\infty}_{\text{loc}}(\omega; \mathbb{R}^3)$ to the system*

$$\partial_\alpha \boldsymbol{\theta} = \boldsymbol{a}_\alpha \quad \textit{in } \omega \quad \textit{and} \quad \boldsymbol{\theta}(y^0) = \boldsymbol{\theta}^0. \tag{5}$$

Since the field F satisfies the system (4), one can see that

$$\partial_\alpha \boldsymbol{a}_\beta = \Gamma_{\alpha\beta}^\sigma \boldsymbol{a}_\sigma + b_{\alpha\beta} \boldsymbol{a}_3 = \Gamma_{\beta\alpha}^\sigma \boldsymbol{a}_\sigma + b_{\beta\alpha} \boldsymbol{a}_3 = \partial_\beta \boldsymbol{a}_\alpha \quad \text{in } \mathcal{D}'(\omega; \mathbb{R}^3).$$

Since $\boldsymbol{a}_\alpha \in W^{1,\infty}_{\text{loc}}(\omega; \mathbb{R}^3)$ and ω is simply connected, we can apply Theorem 1.1 to problem (5) to show that it possesses a solution $\boldsymbol{\theta} \in W^{1,\infty}_{\text{loc}}(\omega; \mathbb{R}^3)$. Since $\partial_\alpha \boldsymbol{\theta} = \boldsymbol{a}_\alpha$, the mapping $\boldsymbol{\theta}$ belongs in fact to $W^{2,\infty}_{\text{loc}}(\omega; \mathbb{R}^3)$.

(iv) *The mapping θ satisfies the relations*

$$a_{\alpha\beta} = \partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} \quad \textit{and} \quad b_{\alpha\beta} = \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} \quad \textit{a.e. in } \omega.$$

Define $a_{\alpha 3}(y) = a_{3\alpha}(y) = 0$ and $a_{33}(y) = 1$ for all $y \in \omega$ and let $\Gamma_{\alpha i}^p$ denote the coefficients of the matrix Γ_α defined by relation (3). Then one can see that $\partial_\alpha a_{ij} = \Gamma_{\alpha i}^p a_{pj} + \Gamma_{\alpha j}^p a_{ip}$. This implies that the functions $X_{ij} : \omega \to \mathbb{R}$ defined by $X_{ij}(y) := \boldsymbol{a}_i(y) \cdot \boldsymbol{a}_j(y) - a_{ij}(y)$ for all $y \in \omega$ satisfy the system

$$\partial_\alpha X_{ij} = \Gamma_{\alpha i}^p X_{pj} + \Gamma_{\alpha j}^p X_{ip} \quad \text{in } L^\infty(\omega),$$
$$X_{ij}(y^0) = 0.$$

Let $\mathcal{A} := \{y \in \omega;\ X_{ij}(y) = 0 \in \mathbb{M}^3\}$. This subset of ω is non-empty (since $y^0 \in \mathcal{A}$) and closed in ω (since the functions $X_{ij} : \omega \to \mathbb{M}^3$ are continuous). But it is also open in ω since an inequality of Poincaré type shows that, for all $y \in \omega$, there exists an open ball $B(x, r) \Subset \omega$ such that $X_{ij} = 0$ over this ball. Since the set $\mathcal{A}$ is non-empty, closed, and open in ω, the conectedness of ω implies that $\mathcal{A} = \omega$. Hence $\boldsymbol{a}_i \cdot \boldsymbol{a}_j = a_{ij}$ in ω. These relations show that the first relation announced in step (iv) holds. They also show that

$$\text{either } \boldsymbol{a}_3(y) = \frac{\boldsymbol{a}_1 \wedge \boldsymbol{a}_2}{|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|}(y) \text{ or } \boldsymbol{a}_3(y) = -\frac{\boldsymbol{a}_1 \wedge \boldsymbol{a}_2}{|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|}(y),\ y \in \omega,\ \text{ and } F^{\mathrm{T}}F = (a_{ij}) \text{ in } \omega.$$

Therefore, $(\det F(y))^2 > 0$, which implies in particular that $\det F(y) \neq 0$ for all $y \in \omega$. Since $\det F(y^0) > 0$ and the function $\det F$ is continuous over the connected set ω, we have $\det F > 0$ over ω. Hence

$$\boldsymbol{a}_3(y) = \frac{\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)}{|\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)|} \quad \text{for all } y \in \omega.$$

On the other hand, step (ii) implies that $\partial_\alpha \boldsymbol{a}_\beta = \Gamma^\sigma_{\alpha\beta}\boldsymbol{a}_\sigma + b_{\alpha\beta}\boldsymbol{a}_3$. Hence

$$b_{\alpha\beta} = \partial_\alpha \boldsymbol{a}_\beta \cdot \boldsymbol{a}_3 = \partial_{\alpha\beta}\boldsymbol{\theta} \cdot \frac{\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}}{|\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}|}.$$

(v) *The mapping $\boldsymbol{\theta}$ is unique up to proper isometries of* $\mathbb{R}^3$.

Let another mapping $\boldsymbol{\phi}$ satisfy the conditions of the theorem. Let $F_0 \in \mathbb{M}^3$ be the matrix whose i-th column is $\boldsymbol{a}_i^0$ and let $E_0 \in \mathbb{M}^3$ be the matrix whose first, second, and third, column are respectively $\partial_1\boldsymbol{\phi}(y^0)$, $\partial_2\boldsymbol{\phi}(y^0)$, and $(\partial_1\boldsymbol{\phi}(y^0) \wedge \partial_2\boldsymbol{\phi}(y^0))/|\partial_1\boldsymbol{\phi}(y^0) \wedge \partial_2\boldsymbol{\phi}(y^0)|$. Define the mapping $\hat{\boldsymbol{\theta}} : \omega \to \mathbb{R}^3$ by letting

$$\hat{\boldsymbol{\theta}}(y) = \boldsymbol{\theta}^0 + Q\big(\boldsymbol{\phi}(y) - \boldsymbol{\phi}\big(y^0\big)\big) \quad \text{for all } y \in \omega, \tag{6}$$

where $Q := F_0E_0^{-1}$. Then one can see that the matrix Q is a proper ortogonal matrix and that the mapping $\hat{\boldsymbol{\theta}}$ satisfies the conditions of the theorem. Therefore

$$\partial_\alpha F = F\Gamma_\alpha \quad \text{and} \quad \partial_\alpha \widehat{F} = \widehat{F}\Gamma_\alpha \quad \text{in } \omega,$$

where F (resp. $\widehat{F}$) is the matrix whose i-th column is $\boldsymbol{a}_i$ (resp. $\hat{\boldsymbol{a}}_i$), with self-explanatory notations.

On the other hand, relation (6) shows that $\hat{\boldsymbol{\theta}}(y^0) = \boldsymbol{\theta}^0$ and $\hat{\boldsymbol{a}}_i(y^0) = \boldsymbol{a}_i^0$. Hence $F(y^0) = \widehat{F}(y^0) = F_0$. Then Theorem 1.1 shows that $F = \widehat{F}$ in Ω, which next implies that $\partial_\alpha\boldsymbol{\theta} = \partial_\alpha\hat{\boldsymbol{\theta}}$. Since $\boldsymbol{\theta}(y^0) = \hat{\boldsymbol{\theta}}(y^0) = \boldsymbol{\theta}^0$ and ω is connected, we finally obtain that $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}$, which means that the mapping $\boldsymbol{\theta}$ satisfying the conditions of the theorem is unique up to proper isometries of $\mathbb{R}^3$.

(vi) The second part of the theorem is a consequence of part (ii) of Theorem 1.1. □

Remark 1. In [2], Hartman and Wintner established the fundamental theorem of surface theory under the assumptions that $a_{\alpha\beta} \in C^1(\omega)$ and $b_{\alpha\beta} \in C^0(\omega)$ together satisfy the Gauss and Codazzi–Mainardi equations in an integral form. Their assumptions are weaker than those used in the classical framework (where $a_{\alpha\beta} \in C^2(\omega)$ and $b_{\alpha\beta} \in C^1(\omega)$) but stronger than those of Theorem 2.1.

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