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Partial Differential Equations

Classification of positive solutions of semilinear elliptic equations

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Abstract

We give a classification of all solutions of a general semilinear PDE in the positive quadrant of $\mathbb{R}^2$. ***To cite this article: J. Busca et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Classification des solutions positives d'une EDP semi-linéaire. Nous donnons une classification de toutes les solutions d'une EDP semi-linéaire générale dans le quadrant positif de $\mathbb{R}^2$. ***Pour citer cet article : J. Busca et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

Resultats principaux

Nous donnons une classification de l'ensemble des solutions bornées de l'EDP semilinéaire

$$\begin{cases} \Delta_{x,y}u = f(u) & \text{dans } \Omega_+, \\ u(x,y) \geqslant 0 & \text{dans } \Omega_+, \\ u_{|\partial\Omega_+} = 0, \end{cases} \tag{1}$$

posée dans le quadrant positif de $\mathbb{R}^2$ noté $\Omega_+ := \{(x,y) \in \mathbb{R}^2,\ x > 0,\ y > 0\}$. Nous supposons ici que $u \in C_b(\overline{\Omega_+})$ et que la non-linéarité f vérifie $f \in C^1(\mathbb{R})$ avec $f(0) = 0$.

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Des résultas de monotonie classiques [2,8] montrent que toute solution $u(x,y)$ de (1), si elle existe, doit être monotone en x et y. Ainsi les limites $\psi_u(y) = \lim_{x\to\infty} u(x,y)$ et $\phi_u(x) = \lim_{y\to\infty} u(x,y)$ sont bien définies et sont solutions de l'EDO

$$\Psi'' = f(\Psi), \quad \Psi(0) = 0, \quad \Psi(z) \geqslant 0, \quad z \geqslant 0. \tag{2}$$

On note que toute solution de cette EDO tend lorsque $z \to \infty$ vers une constante $c \geqslant 0$ telle que $f(c) = 0$ et que, pour un c donné, il existe au plus une solution de ce problème. Cela impose $\psi_u(z) = \phi_u(z) = \Psi_c(z)$ pour un certain $c = c_u > 0$ tel que $f(c) = 0$. On en déduit le comportement à l'infini

$$\lim_{(x,y)\to\infty} \bigl|u(x,y) - \Psi_c(x,y)\bigr| = 0, \quad \text{où } \Psi_c(x,y) := \min\bigl\{\Psi_c(x), \Psi_c(y)\bigr\}. \tag{3}$$

Notre résultat principal établit que tout comportement limite à l'infini est réalisé par une unique solution. Pour cela nous faisons l'hypothèse de non-dégénérescence

$$f'(c) \neq 0 \quad \text{pour tout } c > 0 \text{ pour lequel } f(c) = 0. \tag{4}$$

Notons que l'existence de Ψ_c implique en fait $f'(c) > 0$, voir [4].

Théorème 0.1. *Sous les hypothèses ci-dessus, soit Ψ_c une solution de* (2)*. Alors, il existe une unique solution $u(x,y)$ de* (1) *qui vérifie* (3)*.*

Le corollaire suivant montre que, génériquement, le problème (1) n'admet qu'un nombre fini de solutions.

Corollaire 0.2. *Sous les hypothèses ci-dessus, le problème* (1) *a un nombre fini de solutions.*

1. Introduction and main result

It is well known that positive solutions of semilinear second order elliptic problems have symmetry and monotonicity properties which reflect the symmetry of the operator and of the domain, see, e.g., [9] for the case of bounded domains and [10,2,7,6] for the case of unbounded domains.

In particular, symmetry and monotonicity results for the case of half-spaces have been considered in [3] and analogous results (including the existence and uniqueness of a nontrivial positive solution) for the case of the whole space have been obtained in [10,5,11,6], see also the references therein.

The goal of the present paper is to give a description of all bounded nonnegative solutions of the following elliptic boundary value problem in the two dimensional positive cone $\Omega_+ := \{(x,y) \in \mathbb{R}^2,\ x > 0,\ y > 0\}$

$$\begin{cases} \Delta_{x,y} u = f(u) & \text{in } \Omega_+, \\ u(x,y) \geqslant 0 & \text{in } \Omega_+, \\ u_{|\partial\Omega_+} = 0, \end{cases} \tag{5}$$

where we assume that $u \in C_b(\overline{\Omega_+})$ and $f \in C^1(\mathbb{R})$ with $f(0) = 0$.

It is known (see [2,8]) that, under the above assumptions, every solution $u(x,y)$ of (5) (if it exists) should be monotonic with respect to x and y and, consequently, the limits $\psi_u(y) = \lim_{x\to\infty} u(x,y)$ and $\phi_u(x) = \lim_{y\to\infty} u(x,y)$ are well-defined and are bounded solutions of the ODE

$$\Psi'' = f(\Psi), \quad \Psi(0) = 0, \quad \Psi(z) \geqslant 0, \quad z \geqslant 0. \tag{6}$$

We recall that every solution of (6) stabilizes as $z \to \infty$ to some $c \geqslant 0$ such that $f(c) = 0$ and, for fixed c there exists no more than one solution $\Psi(z) = \Psi_c(z)$ of this problem. This forces $\psi_u(z) = \phi_u(z) = \Psi_c(z)$ for some $c = c_u > 0$ with $f(c) = 0$, this implies the following behaviour at infinity

$$\lim_{(x,y)\to\infty} \bigl|u(x,y) - \Psi_c(x,y)\bigr| = 0, \quad \text{where} \quad \Psi_c(x,y) := \min\bigl\{\Psi_c(x), \Psi_c(y)\bigr\}. \tag{7}$$

Our main result shows that all possible limiting behaviour is achieved by a unique solution. This is established under the nondegeneracy condition

$$f'(c) \neq 0 \quad \text{for all } c > 0 \text{ for which } f(c) = 0. \tag{8}$$

Note that the existence of Ψ_c actually ensures that $f'(c) > 0$, see [4].

Theorem 1.1. *Let the nonlinearity f satisfy the above assumptions and let Ψ_c be a solution of* (6). *Then there exists a unique solution $u(x, y)$ of* (5) *which satisfies* (7).

The following corollary shows that, generically, (5) has only finitely many positive solutions.

Corollary 1.2. *Let the above assumptions hold. Then, problem* (5) *has a finite number of positive bounded solutions.*

2. Sketch of the proof

The proof relies on the following result, which is of independent interest.

Proposition 2.1. *Let u be an arbitrary solution of* (5) *which satisfies* (7). *Then the spectrum of the linearization of* (5) *at u is strictly negative, i.e.,*

$$\sigma\big(\Delta_{x,y} - f'(u)\big) \subset (-\infty, -C_u) \tag{9}$$

for some constant $C_u > 0$, depending only on the solution u.

Proof. Let

$$\Psi_c^M(x, y) = \begin{cases} c, & (x, y) \in [0, M]^2, \\ \Psi_c(x, y), & (x, y) \in \Omega_+ \setminus [0, M]^2, \end{cases} \tag{10}$$

where M is sufficiently large positive number. Then we have that the spectrum of $\Delta_{x,y} - f'(\Psi_c^M(x, y))$ in Ω_+ with Dirichlet boundary conditions is strictly negative, that is,

$$\sigma\big(\Delta_{x,y} - f'\big(\Psi_c^M(x, y)\big), L^2(\Omega_+)\big) \subset (-\infty, -K) \tag{11}$$

for $K > 0$. Indeed in view of the Perron–Frobenius theorem it is a standard fact that

$$\sigma\big(\partial_z^2 - f'\big(\Psi_c(z)\big), L^2(\mathbb{R}_+)\big) \subset (-\infty, -K)$$

for some $K > 0$; one then concludes by the min-max principle and the special form of $\Psi_c(x, y)$.

From (7) and (11), by applying the maximum principle one easily gets the exponential decay

$$\big|u(x, y) - \Psi_c(x, y)\big| \leqslant C\,\mathrm{e}^{-\varepsilon(x+y)}, \quad \forall (x, y) \in \Omega_+ \tag{12}$$

for some $\varepsilon, C > 0$.

Furthermore, observing that $\Delta_{x,y} - f'(u)$ is a compact perturbation of $\Delta_{x,y} - f'(\Psi_c^M)$, one gets that the essential spectrum of $\Delta_{x,y} - f'(u)$ is negative, i.e.,

$$\sigma_{\text{ess}}\big(\Delta_{x,y} - f'(u)\big) \subset (-\infty, -K). \tag{13}$$

We shall now establish (9) by a contradiction argument. Suppose (9) did not hold; in view of (13) this means that there would exist an first eigenvalue $\lambda_0 \geqslant 0$ and an associated eigenvector $v = v(x, y) > 0$ in Ω_+. Using the exponential convergence (12) and (11) one gets

$$\big|v(x, y)\big| \leqslant C_v\,\mathrm{e}^{-\varepsilon(x+y)}, \quad \forall (x, y) \in \Omega_+. \tag{14}$$

We note that the function $v_1(x,y) := \partial_x u(x,y)$ is also strictly positive and satisfies the equation

$$\Delta_{x,y} v_1 - f'\big(u(x,y)\big) v_1 = 0.$$

Multiplying this equation by the eigenvector $v(x,y)$ and integrating over Ω_+, integrating by parts and using the boundary conditions, we derive that

$$\int_0^\infty v_1(0,y)\partial_x v(0,y)\,\mathrm{d}y + \lambda_0 \int_{\Omega_+} v \cdot v_1 \,\mathrm{d}x\,\mathrm{d}y = 0. \tag{15}$$

We now recall that $v_1(x,y) := \partial_x u(x,y) \geqslant 0$, $v(x,y) \geqslant 0$ and $\partial_x v(0,y) > 0$ (due to the strict maximum principle). Consequently, (15) implies that

$$v_1(0,y) := \partial_x u(0,y) \equiv 0. \tag{16}$$

Since $u(0,y) \equiv 0$ due to the boundary conditions, then (16) implies that $u(x,y) \equiv 0$ in Ω_+, due to the unique continuation property for elliptic equations. This contradiction proves the result in Proposition 2.1. □

2.1. *Uniqueness*

Let u_1 and u_2 be two solutions of (5) which satisfy (7). Without loss of generality, we may assume that

$$u_2(x,y) \geqslant u_1(x,y). \tag{17}$$

Indeed, if (17) were not satisfied then, using the sub- and supersolution method (see, e.g., [13]), we may construct a third solution $u_3(x,y)$ such that

$$c \geqslant u_3(x,y) \geqslant \max\big\{u_1(x,y), u_2(x,y)\big\}$$

which does not coincide with u_1 and u_2 and for which (17) is satisfied.

Let us now consider the parabolic boundary value problem in Ω_+

$$\partial_t U = \Delta_{x,y} U - f(U), \quad U|_{\partial\Omega_+} = 0, \quad U|_{t=0} = U_0$$

with the phase space

$$W_0 := \big\{U_0 \in L^\infty(\Omega_+),\ u_1(x,y) \leqslant U_0(x,y) \leqslant u_2(x,y)\big\}. \tag{18}$$

This problem generates a semiflow on the phase space W_0:

$$S_t : W_0 \to W_0, \quad S_t U_0 := U(t)$$

which (according to the general theory, see [1,12]) possesses a global attractor $\mathcal{A}_0 \subset W_0$. Moreover, due to (12) and (18), we have the following Lyapunov function on W_0:

$$L(U_0) := \int_{\Omega_+} \big|\nabla(U_0 - u_1)\big|^2 + 2F_{u_1}(U_0 - u_1, x, y)\,\mathrm{d}x\,\mathrm{d}y,$$

where $F_{u_1}(z,x,y) := \int_0^z f(u_1(x,y)+z) - f(u_1(x,y))\,\mathrm{d}z$.

Thus, the attractor $\mathcal{A}_0$ should consist of heteroclinic orbits to the appropriate equilibria, belonging to W_0 (see [1]), but as proved in Proposition 2.1, all of these equilibria are exponentially stable which is possible only in the case $u_1 \equiv u_2$. Therefore, the uniqueness is also proven and Theorem 1.1 is proven.

2.2. *Proof of Corollary 1.2*

The proof is straightforward if one notices, by integrating (6) (see also the proof of Proposition 4.1 in [4]) , that $F(z) < F(z_0)$ for all $z \in (0, z_0)$, if F denotes a primitive of f.

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