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A kinetic approximation of Hele–Shaw flow

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Abstract

In this Note we consider a fourth order degenerate parabolic equation modeling the evolution of the interface of a spreading droplet. The equation is approximated trough a collisional kinetic equation. This permits to derive numerical approximations that preserves positivity of the solution and the main relevant physical properties. A Monte Carlo application is also shown. *To cite this article: L. Pareschi et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).*

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Résumé

Approximation cinétique des écoulements de Hele–Shaw. Dans cette Note, nous considérons une équation dégénérée du quatrième ordre modélisant l'évolution de l'interface d'une goutte. L'équation est approchée par une équation collisionnelle cinétique. Cela permet de construire des approximations numériques qui préservent la positivité de la solution et ses principales propriétés physiques. Un exemple « Monte-Carlo » est aussi présenté. *Pour citer cet article : L. Pareschi et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).*

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Version française abrégée

Dans cette Note, nous considérons l'équation de diffusion dégénérée du quatrième ordre (1) qui modélise la cellule de Hele–Shaw. Cette équation sera approchée par un modèle collisionnel de Boltzmann (11). La description cinétique présente certains avantages. Entre autres, la positivité de la solution ainsi que la conservation des principales quantités physiques peuvent être facilement établies. Il est alors possible de construire des schémas numériques nouveaux même en utilisant des méthodes de «Monte-Carlo». Il est bien connu, en fait, que la construction de schémas numériques pour (1) qui préservent la positivité de la solution en temps ainsi que ses principales propriétés physiques est délicate, [8,14]. Le processus-limite qui fait passer de l'équation cinétique a l'équation des films fins est analogue à ceux du type «quasi-elastique» pour les gaz granulaires [11] ou « collisions

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rasantes » pour la physique des plasmas [7,13]. Des résultats sont présentés afin d'illustrer la validité de notre approche, cf. Théorèmes 3.1 and 3.2.

1. Introduction

In this Note we consider the fourth order nonlinear degenerate diffusion equation

$$\frac{\partial u}{\partial t} = -(u u_{xxx})_x, \quad x \in \mathbb{R}, \ t > 0, \tag{1}$$

with $u(x, t = 0) = u_0(x) \ge 0$. Eq. (1) is a particular case of the thin film equation

$$\frac{\partial u}{\partial t} = -\left(|u|^n u_{xxx}\right)_x, \quad x \in \mathbb{R}, \ t > 0, \tag{2}$$

where n > 0. The thin film equation (2), derived from a lubrication approximation, models the surface tension dominated motion of thin viscous films and spreading droplets [9]. The different values of n are related to the boundary conditions at the bottom used at the Navier–Stokes level. The most important cases in applications are n = 3, that corresponds to no-slip boundary conditions and n = 1, that corresponds to the so called Hele–Shaw cell [3]. One of the remarkable features of Eq. (2) is that its nonlinearity guarantees the nonnegativity preserving property of the solution [4]. Solutions to (2) are known to preserve mass [1,2]

$$\int_{\mathbb{R}} u(x,t) \, \mathrm{d}x = \int_{\mathbb{R}} u_0(x) \, \mathrm{d}x = M.$$
(3)

Moreover, there is dissipation of surface-tension energy; that is,

$$\int_{\mathbb{R}} u_x^2(x,T) \,\mathrm{d}x + \int_{\mathbb{R} \times [0,T]} u^n u_{xxx}^2(x,t) \,\mathrm{d}x \,\mathrm{d}t = \int_{\mathbb{R}} u_{0x}^2 \,\mathrm{d}x,\tag{4}$$

for all T > 0.

In this Note we will restrict to the case n = 1, usually referred to as Hele–Shaw flow. Eq. (1) satisfies an entropy dissipation principle which highlights similarities with the Boltzmann equation [4]

$$H(u(T)) + \int_{\mathbb{R}\times[0,T]} u_{xx}^2(x,t) \,\mathrm{d}x \,\mathrm{d}t \leqslant H(u_0),\tag{5}$$

where $H(u) = \int_{\mathbb{R}} (u \log u - u) dx$. A further entropy inequality satisfied by the solution to (1) is [1,2]

$$\int_{\mathbb{R}} u^2(x,T) \,\mathrm{d}x + \int_{\mathbb{R} \times [0,T]} u u_{xx}^2 \,\mathrm{d}x \leqslant \int_{\mathbb{R}} u_0^2(x) \,\mathrm{d}x.$$
(6)

Eq. (1) will be approximated by a collisional kinetic model of Boltzmann type. The limiting process that leads from the kinetic equation to the thin film equation is the analogous of the "quasi-elastic" or "grazing collision" limit processes that occur respectively in granular gases [11] and plasma physics [7,13]. The kinetic formulation presents several advantages. Among others it gives positivity of the solution as well as conservation of the main physical properties. This makes it possible to construct new nonnegative numerical schemes even using Monte Carlo methods. It is well known, in fact, that the construction of numerical schemes for (1) that preserve the nonnegativity of the solution in time as well as the other relevant physical properties represents a challenge from a computational point of view [8,14]. We remark that the present kinetic formulation can be generalized to multi-dimensional degenerate fourth order diffusion equations of the type

$$\frac{\partial u}{\partial t} = -\nabla_x \cdot \left(f(u) \nabla_x \Delta_x u \right), \quad x \in \mathbb{R}^3, \ t > 0,$$
(7)

where $f(u) \ge 0$ is the mobility function $(f(u) = |u|^n, n \ge 0)$, in the thin films case).

2. The kinetic formulation

Let $\omega_{\delta}(x)$ be the centered Gaussian density of mass one and second moment equal to $\delta^2 > 0$, that is $\omega_{\delta}(x) = (2\pi)^{-1/2} \delta^{-1} \exp\{-x^2/2\delta^2\}$. For all $t \ge 0$, let v(x, t) be the solution to

$$\frac{\partial v}{\partial t} = -(vv_{xxx} * \omega_{\delta})_{x}, \quad x \in \mathbb{R}, \ t > 0,$$
(8)

corresponding to the initial value u_0 , where f * g denotes the convolution of the two functions f and g. By virtue of (3), without loss of generality, we can fix the mass M = 1.

Let us set $A_{\delta}(x) = -(\omega_{\delta}(x))_{xxx}$. Using properties of the convolution operator one shows that Eq. (8) can be equivalently written as

$$\frac{\partial u}{\partial t} = Q_{\delta}(u) = \frac{\partial}{\partial x} \left(u(x) \int_{\mathbb{R}} A_{\delta}(y) u(x-y) \, \mathrm{d}y \right).$$
(9)

Eq. (9) has the same structure of some dissipative equations studied recently in connection with the cooling of granular gases [11]. Having this analogy in mind, we consider a suitable approximation of (9), which has the structure of a Boltzmann-like equation. Let us mention that a similar idea for the Landau–Fokker–Planck equation has been used recently in [5]. With this aim, given any constant α , with $0 \le \alpha \le 1$, we approximate the derivative in (9) by introducing a small parameter ε with the same sign of A_{δ} ,

$$\varepsilon = h \frac{A_{\delta}}{|A_{\delta}|^{\alpha}}, \quad A_{\delta} \neq 0, \qquad \varepsilon = 0, \quad A_{\delta} = 0.$$
 (10)

In this way we obtain the Boltzmann-like equation

$$\frac{\partial u}{\partial t} = Q_{\delta}^{(\alpha)}(u) = \frac{1}{h} \int_{\mathbb{R}} \left| A_{\delta}(y) \right|^{\alpha} \left(u(x')u(x'-y) - u(x)u(x-y) \right) \mathrm{d}y, \tag{11}$$

where the "pre-collisional" velocity x' is given by $x' = x + \varepsilon$.

We will mainly consider here the two limit cases $\alpha = 0$ and $\alpha = 1$. In the former case, using the mass conservation property, we get

$$Q_{\delta}^{(0)}(u) = \frac{1}{h} \int_{\mathbb{R}} u(x')u(x'-y) - \frac{1}{h}u(x) \int_{\mathbb{R}} u(y) \, \mathrm{d}y = \frac{1}{h} \int_{\mathbb{R}} u(x')u(x'-y) - \frac{1}{h}u(x).$$
(12)

Note that Eq. (12) has a constant rate function, which is typical of the Boltzmann equation for Maxwell molecules [6]. For this reason, from now on we will refer to it as the Maxwellian approximation of Eq. (9). Following standard arguments of kinetic theory [6] it is easy to show that the collisional kinetic models (11) satisfy conservation of mass and momentum as well as positivity of the solution.

3. Consistency

For $m \ge 1$, let $C_0^m(\mathbb{R})$ be the set of *m*-times compactly supported continuously differentiable functions, endowed with its natural norm $\|\varphi\|_m \equiv \sum_{k=0}^m \max_{x \in \mathbb{R}} |\varphi^{(k)}|$. Given two probability density functions f(x) and $g(x), x \in \mathbb{R}$, let

$$\|f - g\|_m^* = \sup\left\{ \left| \int \varphi(x) \left(f(x) - g(x) \right) dx \right|, \ \varphi \in C_0^m, \ \|\varphi\|_m \le 1 \right\}.$$

$$\tag{13}$$

The above formula defines a distance which metrizes the weak-* topology on $P_s(\mathbb{R})$, s > 0, namely the class of all probability distributions F on \mathbb{R} , such that $\int_{\mathbb{R}} |x|^s dF(x) < \infty$. This distance has been extensively used in [12], where precise connections with other more known metrics have been established. We prove

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Theorem 3.1. We are given a nonnegative, initial condition $u_0 \in L^1(\mathbb{R}) \cap H^1(\mathbb{R})$ with unit mass and $x^2 u_0 \in L^1(\mathbb{R})$. Then, there exists a strong solution u(t, x) to the Cauchy problem (1) such that, for all T > 0, the distance $\|uu_{xxx}(\cdot, T) - uu_{xxx} * \omega_{\delta}(\cdot, T)\|_{3}^{*}$ tends to zero as $\delta \to 0$, and the following bounds holds

$$\left\| uu_{xxx}(\cdot,T) - uu_{xxx} * \omega_{\delta}(\cdot,T) \right\|_{3}^{*} \leq \delta C(u_{0}).$$
⁽¹⁴⁾

Proof. Let $\Omega_T = \mathbb{R} \times [0, T], \varphi \in C_0^3, \|\varphi\|_3 \leq 1$. Integrating by parts,

$$\left|\int_{\Omega_T} \varphi(x) \left(u(u_{xxx} - u_{xxx} * \omega_{\delta}) \right)_x \right| = \left| \int_{\Omega_T} (\varphi_x u)_{xx} (u_x - u_x * \omega_{\delta}) \right|.$$
(15)

Since $\|\varphi\|_3 \leq 1$, $|(\varphi_x u)_{xx}| \leq |u| + 2|u_x| + |u_{xx}|$. Hence, by the Cauchy–Schwarz inequality

$$\left| \int_{\Omega_T} (\varphi_x u)_{xx} (u_x - u_x * \omega_\delta) \right| \leq \left[\left(\int_{\Omega_T} u^2 \right)^{1/2} + 2 \left(\int_{\Omega_T} u^2_x \right)^{1/2} + \left(\int_{\Omega_T} u^2_{xx} \right)^{1/2} \right] \left(\int_{\Omega_T} (u_x - u_x * \omega_\delta)^2 \right)^{1/2}.$$

By (4) and (6)

$$\left(\int_{\Omega_T} u^2\right)^{1/2} + 2\left(\int_{\Omega_T} u_x^2\right)^{1/2} \leqslant T^{1/2} \left[\left(\int_{\mathbb{R}} u_0^2\right)^{1/2} + 2\left(\int_{\mathbb{R}} u_{0,x}^2\right)^{1/2}\right].$$

Moreover, from the entropy decay 5 and from Gibbs lemma (that states that the Gaussian is the function that maximizes entropy with given moments) [6], $(\int_{\Omega_T} u_{xx}^2)^{1/2} \leq H(u_0) - H(u(T)) \leq H(u_0) - H(\omega_{\sigma_T})$, where $\sigma_T = \sqrt{\int_{\mathbb{R}} x^2 u_T}$. Since $\sigma_T^2 \leq \sigma_0^2 + \frac{3}{2} \int_{\mathbb{R}} u_{0x}^2$, $\left(\int_{\Omega_T} u_{xx}^2\right)^{1/2} \leq \left(H(u_0) + \frac{1}{2} \log 2\pi \left(\sigma_0 + \frac{3}{2} \int_{\mathbb{R}} u_{0x}^2\right)\right)^{1/2}$. (16)

Let \hat{f} denote the Fourier transform of f. By Parseval's identity, given any $f \in H^1(\mathbb{R})$,

$$\int_{\mathbb{R}} (f - f * \omega_{\delta})^2 = \frac{1}{2\pi} \int_{\mathbb{R}} (\hat{f} - \hat{f} e^{-\delta^2 \xi^2})^2 \leq \frac{\delta^2}{2\pi} \int_{\mathbb{R}} |\xi \hat{f}|^2 \frac{1 - e^{-\delta^2 \xi^2}}{\delta^2 |\xi|^2} \leq \frac{\delta^2}{2\pi} \int_{\mathbb{R}} |\xi \hat{f}|^2 = \delta^2 \int_{\mathbb{R}} (f_x)^2.$$
(17)

Hence, $(\int_{\Omega_T} (u_x - u_x * \omega_\delta)^2)^{1/2} \leq \delta (\int_{\Omega_T} (u_{xx})^2)^{1/2}$, and by (16) we have

$$\left(\int_{\Omega_T} \left(u_x - u_x * \omega_\delta\right)^2\right)^{1/2} \leqslant \delta \left(\int_{\mathbb{R}} H_0(u_0) + \frac{1}{2}\log 2\pi \left(\sigma_0^2 + \frac{3}{2}\int_{\mathbb{R}} u_{0x}^2\right)\right)^{1/2}.$$
(18)

This concludes the proof. \Box

The following theorem gives similar estimates for the distance between the operator defined by (9) and the corresponding Boltzmann-like collision operators given in (11).

Theorem 3.2. We are given a nonnegative, initial condition $u_0 \in L^1(\mathbb{R}) \cap H^1(\mathbb{R})$ with unit mass. Then, there exists a strong solution u(t, x) to the Cauchy problem (1) such that, for all T > 0, the distance $\|\delta^{7-4\alpha}Q_{\delta}(u(\cdot, T)) - \delta^{7-4\alpha}Q_{\delta}^{(\alpha)}(u(\cdot, T))\|_2^*$ tends to zero as $h \to 0$, and the following bound holds

$$\left\|\delta^{7-4\alpha}Q_{\delta}\left(u(\cdot,T)\right) - \delta^{7-4\alpha}Q_{\delta}^{(\alpha)}\left(u(\cdot,T)\right)\right\|_{2}^{*} \leqslant \frac{1}{2}h\int_{\mathbb{R}}\left|A_{1}(x)\right|^{2-\alpha}\mathrm{d}x\int_{\mathbb{R}}u_{0}^{2}\mathrm{d}x.$$
(19)

Proof. Let $\varphi \in C_0^2$, $\|\varphi\|_2 \leq 1$. Integrating by parts we get

$$\int_{\mathbb{R}} \varphi(x) Q_{\delta}(u(x)) dx = \int_{\mathbb{R}} \varphi(x) \frac{\partial}{\partial x} \left(u(x) \int_{\mathbb{R}} A_{\delta}(y) u(x-y) dy \right) dx$$
$$= -\int_{\mathbb{R}} \int_{\mathbb{R}} A_{\delta}(y) \varphi'(x) u(x) u(x-y) dy dx.$$
(20)

On the other hand

$$\int_{\mathbb{R}} \varphi(x) Q_{\delta}^{(\alpha)}(u(x)) dx = \int_{\mathbb{R}} \varphi(x) \int_{\mathbb{R}} |A_{\delta}(y)|^{\alpha} \frac{1}{h} (u(x')u(x'-y) - u(x)u(x-y)) dy dx$$
$$= \int_{\mathbb{R}} \int_{\mathbb{R}} |A_{\delta}(y)|^{\alpha} \frac{1}{h} \left(\varphi \left(x - h \frac{A_{\delta}}{|A_{\delta}|^{\alpha}} \right) - \varphi(x) \right) u(x)u(x-y) dy dx$$
$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \left(-\varphi'(x)A_{\delta}(y) + \frac{1}{2}\varphi''(\bar{x})|A_{\delta}(y)|^{2-\alpha} \right) u(x)u(x-y) dy dx.$$
(21)

Hence, since $\|\varphi\|_2 \leq 1$, which implies $|\varphi''(\bar{x})| \leq 1$,

$$\left| \int_{\mathbb{R}} \varphi(x) \left(Q_{\delta} \left(u(\cdot, T) \right) - Q_{\delta}^{(\alpha)} \left(u(\cdot, T) \right) \right) \right| \leq \frac{1}{2} h \iint_{\mathbb{R}} \left| A_{\delta}(y) \right|^{2-\alpha} u(x) u(x-y) \, \mathrm{d}y \, \mathrm{d}x.$$
(22)

To conclude, consider that

$$A_{\delta}(x) = \frac{1}{\delta^3} \left(\left(\frac{x}{\delta}\right)^3 - 3\frac{x}{\delta} \right) \frac{1}{(2\pi)^{1/2}\delta} \exp\left\{ -\frac{x^2}{2\delta^2} \right\},\tag{23}$$

is such that $|A_{\delta}(y)|^{2-\alpha} = B_{\delta}(y)/\delta^{7-4\alpha}$, with $0 \leq B_{\delta}(y) \in L^1(\mathbb{R})$, and $||B_{\delta}||_{L^1(\mathbb{R})} = ||A_1^{2-\alpha}||_{L^1(\mathbb{R})}$. By Cauchy–Schwarz inequality, and using the property of monotonicity in time of $\int u^2$, we obtain bound (19).

4. A Monte Carlo scheme

The simplest way to solve numerically the collision integrals (11) is by direct simulation Monte Carlo (DSMC). We compare the numerical results obtained with our kinetic approximation with the similarity solution

$$u(x,t) = \frac{1}{120(t+\tau)^{1/5}} \left[\omega^2 - \frac{x^2}{(t+\tau)^{2/5}} \right]_+^2,$$
(24)

where $[\cdot]_+$ denotes the positive part. We have chosen as in [8] $\omega = 2, \tau = 4^{-5}, x \in [0, 1]$.

For the details of the Monte Carlo method we refer to [10]. We remark that the Monte Carlo method has been built so that positivity of the solution as well as mass and momentum are preserved at a discrete level. The results reported in Fig. 1 have been obtained with $N = 50\ 000$ particles, $\delta = 0.1$ at time t = 0.005. The value of h used in the computations are $h = 2 \times 10^{-6}$ for $\alpha = 0$ and $h = 5 \times 10^{-3}$ for $\alpha = 1$.

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Fig. 1. Solution at t = 0.005 for $\alpha = 0$ (left) and $\alpha = 1$ (right). The dotted line represents the initial datum.

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