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COMPTES RENDUS MATHEMATIQUE

Probability Theory

Exponential divergence estimates and heat kernel tail

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Abstract

We obtain a lower bound for the density of a real random variable on the Wiener space under an exponential moment condition of the divergence. We apply this result to the solution of a non-linear SDE. ***To cite this article: E. Nualart, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Estimations exponentielles de divergence et queue du noyau de la chaleur. Nous obtenons une minoration exponentielle de la densité d'une variable aléatoire réelle dans l'espace de Wiener sous une condition portant sur le moment exponentiel de la divergence. Nous appliquons ce résultat à la solution d'une EDS non-linéaire. ***Pour citer cet article : E. Nualart, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

1. Minoration of density in terms of divergence

Consider a non-degenerate $\mathbb{R}$-valued random variable F on the Wiener space $\mathcal{W}$, such that $F \in D^{3,\infty}(\mathcal{W})$. Let A be a covering vector field of $\frac{\partial}{\partial\xi}$ for F, that is, a process on $\mathcal{W}$ such that, formally, $\langle DF, A\rangle = \frac{\partial}{\partial\xi}$. This means that for $\phi : \mathbb{R} \mapsto \mathbb{R}$ smooth,

$$\mathbb{E}\left[\frac{\partial\phi}{\partial\xi}(F)\right] = \mathbb{E}\big[\phi(F)\delta(A)\big],$$

where $\delta(A)$ denotes the divergence of A, which is supposed to exist.

Theorem 1.1. *Assume that there exists $\gamma \geqslant 1$ and $c > 0$ such that*

$$\mathbb{E}\big[\exp\big(c\big|\delta(A)\big|^{\gamma}\big)\big] < \infty.$$

Then the law of F has a continuously differentiable density $p(\xi)$ which, if $\gamma > 1$, satisfies the minoration

$$p(\xi) > c_\gamma \exp\big(-c_\gamma |\xi|^{\gamma/(\gamma-1)}\big), \quad \textit{for all } \xi \in \mathbb{R}. \tag{1}$$

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If $\gamma = 1$, *we have* $p(\xi) > c\exp(-c\exp(\xi))$.

Proof. Let $\phi : \mathbb{R} \mapsto \mathbb{R}$ be smooth. By the integration by parts formula in $\mathbb{R}$,

$$\int_{\mathbb{R}} \phi(\xi)\frac{\partial p}{\partial \xi}(\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}} \phi(\xi)\,\mathbb{E}\big[\delta(A)|F=\xi\big]\,p(\xi)\,\mathrm{d}\xi.$$

Therefore, $v'(\xi) := \frac{p'}{p}(\xi) = \mathbb{E}[\delta(A)|F=\xi]$.

Consider the function $\psi : \eta \mapsto \exp(c\,\eta^{\gamma}),\ \eta > 0$. By Jensen's inequality,

$$\mathbb{E}\big[\psi\big(\big|v'(F)\big|\big)\big] \leqslant \mathbb{E}\big[\psi\big(\big|\delta(A)\big|\big)\big]. \tag{2}$$

Introduce the increasing function $w(\xi) = \int_a^{\xi} |v'(u)|\,\mathrm{d}u,\ \xi > a$, where a is a point in the interior of the support of the law of F. In particular, $p(a) > 0$. We assume without loss of generality that $a = 0$. The case $\xi < 0$ follows similarly.

Note that

$$p(\xi) \geqslant p(0)\exp\big(-w(\xi)\big). \tag{3}$$

By (2) and (3), we obtain

$$\int_0^{+\infty} \exp\big(c(w')^{\gamma}(\xi) - w(\xi)\big)\,\mathrm{d}\xi \leqslant \frac{1}{p(0)}\mathbb{E}\big[\exp\big(c\big|\delta(A)\big|^{\gamma}\big)\big] < \infty.$$

The resolution of $c(w')^{\gamma}(\xi) - w(\xi) = 0$ shows that the exponent appearing in (1) cannot be improved.

We now define the increasing sequence $\alpha_n = \inf\{\xi\colon\ w(\xi) \geqslant n\}$. Consider the functional

$$J_n(w) := \int_{\alpha_n}^{\alpha_{n+1}} \exp\big(c(w')^{\gamma}(\xi)\big)\,\mathrm{d}\xi.$$

Lemma 1.2 (Variational Lemma). *The minimum of* $J_n(w)$ *among all increasing and continuously differentiable functions* $w : [\alpha_n, \alpha_{n+1}] \mapsto \mathbb{R}$ *such that* $\int_{\alpha_n}^{\alpha_{n+1}} w'(\xi)\,\mathrm{d}\xi = 1$, *is reached for* $w(\xi) = l_n^{-1}\xi$, *where* $l_n := \alpha_{n+1} - \alpha_n$.

Proof. We apply the Lagrange multipliers method looking for the absolute minimum of $J_n(w) - \lambda\int_{\alpha_n}^{\alpha_{n+1}} w'(\xi)\,\mathrm{d}\xi$, where λ is a constant. Denoting δ a variational of w', we must have at the minimum

$$\int_{\alpha_n}^{\alpha_{n+1}} \big(\gamma c\big(w'(\xi)\big)^{\gamma-1}\exp\big(c(w')^{\gamma}(\xi)\big) - \lambda\big)\delta\,\mathrm{d}\xi = 0, \quad \text{for all } \delta.$$

Therefore, $\gamma c(w'(\xi))^{\gamma-1}\exp(c(w')^{\gamma}(\xi)) = \lambda$, which implies $w'(\xi)$ equal a constant determined by the condition $\int_{\alpha_n}^{\alpha_{n+1}} w'(\xi)\,\mathrm{d}\xi = 1$.

As a consequence,

$$\int_{\alpha_n}^{\alpha_{n+1}} \exp\big(c(w')^{\gamma}(\xi) - w(\xi)\big)\,\mathrm{d}\xi \geqslant \exp\big(-(n+1)\big)l_n\exp\big(cl_n^{-\gamma}\big).$$

As

$$\sum_{n=0}^{\infty}\int_{\alpha_n}^{\alpha_{n+1}} \exp\big(c(w')^{\gamma}(\xi) - w(\xi)\big)\,\mathrm{d}\xi < \infty,$$

we must have $\exp(-(n+1))l_n \exp(cl_n^{-\gamma}) \to 0$, as $n \to \infty$. This implies that for all $n \geqslant n_0$, $(n+1) - cl_n^{-\gamma} > 0$, for some constant $c > 0$. In particular, for $q > n_0$,

$$\sum_{n=n_0}^{q} l_n = \alpha_{q+1} - \alpha_{n_0} > \sum_{n=n_0}^{q} \frac{c_\gamma}{(n+1)^{1/\gamma}} > \int_{n_0+1}^{q+1} \frac{c_\gamma}{x^{1/\gamma}}\,\mathrm{d}x.$$

Note that as $\xi < \alpha_{q+1},\ w(\xi) < q+1$. Therefore, the desired estimate follows using (3).

2. Stochastic differential equations

Let $x = (x(t),\ t \in [0,T])$ be the solution of the following SDE:

$$\mathrm{d}x(t) = a\big(x(t)\big)\,\mathrm{d}W(t) + b\big(x(t)\big)\,\mathrm{d}t, \quad x(0) = 0,$$

where W is a Brownian motion and a and b are $C^3(\mathbb{R})$-functions with bounded derivatives.

The derivative $D_r(x(t))$ satisfies the following linear SDE:

$$D_r\big(x(t)\big) = a\big(x(r)\big) + \int_r^t D_r\big(a\big(x(s)\big)\big)\,\mathrm{d}W(s) + \int_r^t D_r\big(b\big(x(s)\big)\big)\,\mathrm{d}s.$$

The Jacobian $J = (J(t),\ t \in [0,T])$ is the solution of the linearized SDE

$$\mathrm{d}J(t) = \big(a'\big(x(t)\big)\,\mathrm{d}W(t) + b'\big(x(t)\big)\,\mathrm{d}t\big)J(t), \quad J(0) = 1.$$

One can easily check that $D_r(x(t)) = J(t)J^{-1}(r)a(x(r))$.

We associate to J the rescaled variation defined as $z(t) = J(t)/a(x(t))$.

Proposition 2.1 [1]. *The rescaled variation is a differentiable function of t and*

$$z(t) = z(0)\exp\left(\int_0^t \lambda\big(x(s)\big)\,\mathrm{d}s\right), \quad \textit{where } \lambda = b' - \frac{a'}{a}b - \frac{1}{2}a''a \textit{ is called the feedback effect rate.}$$

Therefore,

$$D_r\big(x(t)\big) = a\big(x(t)\big)\exp\left(\int_r^t \lambda\big(x(s)\big)\,\mathrm{d}s\right).$$

We now fix $t > 0$ and define the process

$$u(r) = \frac{1}{t}\frac{1}{a(x(t))}\exp\left(-\int_r^t \lambda\big(x(s)\big)\,\mathrm{d}s\right), \quad r \leqslant t.$$

Proposition 2.2. *For any function $\phi \in C_b^1(\mathbb{R})$ and for $t > 0$, the following formula holds*:

$$\mathbb{E}\big[\phi'\big(x(t)\big)\big] = \mathbb{E}\big[\phi\big(x(t)\big)\delta(u)\big].$$

Proof. By the duality relation between D and δ, and the definition of u,

$$\mathbb{E}\big[\phi\big(x(t)\big)\delta(u)\big] = \mathbb{E}\left[\int_0^t \phi'\big(x(t)\big) D_r\big(x(t)\big) u(r)\,\mathrm{d}r\right] = \mathbb{E}\big[\phi'\big(x(t)\big)\big].$$

Theorem 2.3. *Assume that $\lambda,\ \lambda' a,\ a'$ are bounded and $|a(\xi)| \geqslant c > 0$, for all $\xi \in \mathbb{R}$.*
Then for all $t \geqslant t_0 > 0,\ x(t)$ admits a continuously differentiable density such that

$$p_t(\xi) > c\exp\big(-c|\xi|^2\big), \quad \text{for all } \xi \in \mathbb{R}, \tag{4}$$

where c is a constant depending on t and the bounds on the coefficients.

Proof. Fix $t \geqslant t_0 > 0$ and define the random variable and the adapted process

$$v(t) = \frac{1}{t}\frac{1}{a(x(t))}\exp\left(-\int_0^t \lambda\big(x(s)\big)\,\mathrm{d}s\right), \qquad w(r) = \exp\left(\int_0^r \lambda\big(x(s)\big)\,\mathrm{d}s\right), \quad r \leqslant t.$$

By the properties of δ,

$$\delta(u) = v(t)\int_0^t w(r)\,\mathrm{d}W(r) - \int_0^t D_r\big(v(t)\big) w(r)\,\mathrm{d}r.$$

We now compute the derivative $D_r(v(t))$:

$$D_r\big(v(t)\big) = \frac{1}{t}\frac{1}{a(x(t))} D_r \exp\left(-\int_0^t \lambda\big(x(s)\big)\,\mathrm{d}s\right) + D_r\left(\frac{1}{t}\frac{1}{a(x(t))}\right)\exp\left(-\int_0^t \lambda\big(x(s)\big)\,\mathrm{d}s\right),$$

where

$$D_r\left(\frac{1}{t}\frac{1}{a(x(t))}\right) = \frac{1}{t}\frac{a'(x(t))}{a(x(t))}\exp\left(\int_r^t \lambda\big(x(s)\big)\,\mathrm{d}s\right), \quad \text{and}$$

$$D_r \exp\left(-\int_0^t \lambda\big(x(s)\big)\,\mathrm{d}s\right) = -\exp\left(-\int_0^t \lambda\big(x(s)\big)\,\mathrm{d}s\right)\int_0^t \lambda'\big(x(s)\big) a\big(x(s)\big)\exp\left(\int_r^s \lambda\big(x(s)\big)\,\mathrm{d}s\right)\mathrm{d}s.$$

By the exponential martingale inequality, there exists $c > 0$ depending on t such that $\mathbb{E}[\exp(c|\delta(u)|^2)] < \infty$. The conclusion follows from Proposition 2.2 and Theorem 1.1.

Corollary 2.4. *Suppose $a = 1$. Then if $b',\ b''$ are bounded, the minoration* (4) *holds.*

Remark 1. In one dimension by a change of variables we can always reduce to the case $a = 1$.

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References

[1] E. Barucci, P. Malliavin, M.E. Mancino, R. Renò, A. Thalmaier, The price-volatility feedback rate: an implementable mathematical indicator of market stability, Math. Finance 13 (2003) 17–35.