Available online at www.sciencedirect.com
SCIENCE DIRECT®
C. R. Acad. Sci. Paris, Ser. I 338 (2004) 781–786
COMPTES RENDUS MATHEMATIQUE

Algebraic Geometry

A p-adic proof of Hodge symmetry for threefolds

Kirti Joshi

Department of Mathematics, University of Arizona, 617 N Santa Rita, Tucson, AZ 85721-0089, USA

Received 24 October 2002; accepted after revision 12 November 2003

Presented by Jean-Pierre Serre

Dedicated to the memory of Pū. La. Deshpande

Abstract

We give a p-adic proof of Hodge symmetry for smooth and projective varieties of dimension three over the field of complex numbers. ***To cite this article: K. Joshi, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Une preuve p-adique de la symétrie de Hodge pour les variétés de dimension 3. Nous donnons une preuve p-adique de la symétrie de Hodge pour une variété complexe, projective et lisse de dimension trois. ***Pour citer cet article : K. Joshi, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

Dans cette Note nous donnons une preuve p-adique de la symétrie de Hodge pour les variétés projectives, lisses de dimension au plus trois (Théorème 3.1). Notre approche est basée sur l'observation suivante : d'abord on observe que l'analogue p-adique de la symétrie de Hodge est vraie pour une variété projective et lisse sur un corps parfait de caractéristique $p > 0$ (Théorème 3.2) où les nombres de Hodge sont remplacés par les invariantes p-adiques introduites par Ekedahl, appelées nombres de Hodge–Witt. La symétrie de Hodge–Witt pour des variétés déjà décrites était montrée par Ekedahl, en utilisant son théorème de dualité. Ensuite, on a utilisé un autre résultat de Ekedahl qui donne l'égalité des nombres de Hodge–Witt et ceux de Hodge sous certaines conditions. La dégénération de la suite spectrale de Hodge–de Rham en caractéristique $p > 0$ implique que ces conditions sont satisfaites. Pour déduire le Théorème 3.1 du Théorème 3.2 on utilise le « spreading out argument » standard et le critère de Deligne–Illusie. Dans la Remarque 2 nous démontrons que l'hypothèse que la cohmologie cristalline de X soit sans torsion est nécessaire.

E-mail address: kirti@math.arizona.edu (K. Joshi).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2003.11.038

1. Introduction

Let $X/\mathbb{C}$ be a smooth projective variety. For $p, q \geqslant 0$, let $h^{p,q} = \dim H^q(X, \Omega^p_{X/\mathbb{C}})$. By the Hodge decomposition theorem (see [7, Chapter 0, Section 7]) the spaces $H^q(X, \Omega^p_{X/\mathbb{C}})$, $H^p(X, \Omega^q_{X/\mathbb{C}})$ are complex conjugates and hence for all $p, q \geqslant 0$ we have:

$$h^{p,q} = h^{q,p}. \tag{1}$$

In this Note we give a p-adic proof of (1) when $X/\mathbb{C}$ is a smooth projective variety of dimension at most three (see Theorem 3.1). Our approach is based on the following observation: One first notes that a p-adic analogue of (1) holds when $X/\mathbb{C}$ is replaced by a smooth, projective variety of dimension at most three over a perfect field of characteristic $p > 0$ (see Theorem 3.2) and Hodge numbers are replaced by p-adic invariants introduced by Ekedahl [5, IV 3.1] called Hodge–Witt numbers. These invariants take into account the torsion in the slope spectral sequence as well as the slopes of Frobenius in the crystalline cohomology of the variety. Hodge–Witt symmetry for such varieties was proved by Ekedahl (see [5, VI 3.3 (ii)]) using his duality theorem.

Next one appeals to another result of Ekedahl (see [5, IV, 3.3.1]) which guarantees the equality of Hodge–Witt numbers and the Hodge numbers under suitable circumstances. That these required conditions are met is a simple consequence of the degeneration of Hodge–de Rham spectral sequence in characteristic p (which is a hypothesis in Theorem 3.2). To deduce Theorem 3.1 from Theorem 3.2 we use a standard “spreading out argument” and the Deligne–Illusie criterion (see [2, Corollary 2.4]) for the deneration of the Hodge to de Rham spectral sequence. In Remark 2 we show that in Theorem 3.2 the hypothesis that the crystalline cohomology of X be torsion free cannot be relaxed.

The restriction on dimension in Theorems 3.1, 3.2 arises because Hodge–Witt symmetry is not known to hold in all dimensions. However, Ekedahl (see [5, VI, 3.2 (ii)]) gave a necessary and sufficient condition for Hodge–Witt symmetry to hold (in any dimension). This condition is given in terms of a certain equality of domino numbers for the slope spectral sequence (see [11]). We conjecture that Hodge–Witt symmetry (and hence Ekedahl’s necessary and sufficient conditions) holds under a fairly mild hypothesis on X (see Conjecture 1). As Hodge symmetry is true over complex numbers one can turn our proof of Theorem 3.1 around to deduce the validity of Conjecture 1 under the slightly stronger hypothesis that X admits a smooth lifting to Witt vectors. This is the content of Proposition 3.1.

2. Hodge–Witt symmetry

To keep this Note brief, we will refer to [9,11] and [3–5] for notations and basic results. In particular we do not recall the notion of dominos here but make use of it. All the properties of dominos we need here are conveniently summarized in Illusie’s survey [10]. In this section X/k is a smooth projective variety over a perfect field of characteristic $p > 0$. Let $H^j(X, W\Omega^i_X)$ be the Hodge–Witt cohomology groups of X. Let $T^{i,j}$ be the dimension of the domino (see [10, p. 42]) associated to the differential

$$H^j\big(X, W\Omega^i_X\big) \to H^j\big(X, W\Omega^{i+1}_X\big).$$

Let $m^{i,j}$ be the slope numbers (see [5, 0, 6.1 and 6.2 (ii)] or [10, p. 64]) associated to the slopes of Frobenius on the crystalline cohomology of X. We recall the definition here for the reader’s convenience. Let for any rational number λ, let $h^n_{\mathrm{cris},\lambda}$ be the dimension (= multiplicity) of the slope λ in $H^n_{\mathrm{cris}}(X/W)$. Then by definition of $m^{i,j}$ we have

$$m^{i,j} = \sum_{\lambda \in [i,i+1)} (i + 1 - \lambda) h^{i+j}_{\mathrm{cris},\lambda} + \sum_{\lambda \in [i-1,i)} (\lambda - i + 1) h^{i+j}_{\mathrm{cris},\lambda}. \tag{2}$$

Then the Hodge–Witt numbers of X (see [10, p. 64]), denoted $h^{i,j}_W$, are defined to be

$$h^{i,j}_W = m^{i,j} + T^{i,j} - 2T^{i-1,j+1} + T^{i-2,j+2}. \tag{3}$$

Note that by [11, I, 2.18.1] $T^{i,j}$ is zero if the corresponding differential of the slope spectral sequence is zero.

The following symmetry of slope numbers is a consequence of [1] and [14, Theorem 1] and is due to Ekedahl (see [5, VI, 3.1 (ii)]).

Lemma 2.1. *For any smooth projective variety X/k over a perfect field k of characteristic p and for all i, j we have*

$$m^{i,j} = m^{j,i}. \tag{4}$$

We will also need the following elementary lemma due to Ekedahl (see [5, VI, 3.3 (ii)]).

Lemma 2.2. *Let X/k be a smooth projective variety over a perfect field k of characteristic $p > 0$. Then*

$$h_W^{0,1} = h_W^{1,0}, \tag{5}$$

$$h_W^{0,2} = h_W^{2,0}. \tag{6}$$

Proof. From the definition of $h_W^{i,j}$ (see (3)) and the fact that the domino number $T^{i,j} = 0$ if one of i, j is negative, we see that we have

$$h_W^{0,1} = m^{0,1} + T^{0,1}, \tag{7}$$

$$h_W^{1,0} = m^{1,0} + T^{1,0} - 2T^{0,1}, \tag{8}$$

$$h_W^{0,2} = m^{0,2} + T^{0,2}, \tag{9}$$

$$h_W^{2,0} = m^{2,0} + T^{2,0} - 2T^{1,1} + T^{0,2}. \tag{10}$$

Now by Lemma 2.1 it suffices to prove that $T^{0,1}$, $T^{1,1}$ and $T^{1,0}$, $T^{2,0}$ are zero. The vanishing of the first two is a consequence of the fact that $H^1(X, W(\mathcal{O}_X))$, $H^1(X, W\Omega_X^1)$ are finite type over W (see [11, II, 3.11]) and the vanishing of the remaining two is a consequence of the fact that $H^0(X, W\Omega_X^1)$, $H^0(X, W\Omega_X^2)$ are of finite type over W, which is proved in [9, 2.18, p. 614]. $\square$

Remark 1. In [5, VI, 3.3] Ekedahl has shown that $h_W^{i,j} = h_W^{j,i}$ for any smooth projective threefold X/k and $i, j \geqslant 0$. The proof of the assertion $h_W^{i,3-i} = h_W^{3-i,i}$ uses Ekedahl's duality theorem (see [3]). We note here that we do not use Ekedahl's duality in the proof of Theorem 3.2 given below but use the two lemmas above. We also note that Ekedahl's duality is not sufficient to prove Hodge–Witt symmetry in higher dimensions.

3. Hodge symmetry for threefolds

Theorem 3.1. *Let $X/\mathbb{C}$ be a smooth projective variety over complex numbers of dimension at most three. Then hodge symmetry* (1) *holds for X.*

This theorem will follow from the following more general assertion.

Theorem 3.2. *Let X/k be a smooth projective variety of dimension at most three over a perfect field k of characteristic $p > 0$. Assume that the Hodge to de Rham spectral sequence of X degenerates and the crystalline cohomology of X is torsion free. Then Hodge symmetry* (1) *holds for Hodge numbers of X/k.*

Proof. Observe that the equality $h^{i,\dim(X)-i} = h^{\dim(X)-i,i}$ is a trivial consequence of Serre duality (see [8, III, Theorem 7.6]). So we have to prove $h^{i,j} = h^{j,i}$ when $i + j \leqslant \dim(X) - 1$. This already proves the result for

$\dim(X) = 1$. Now observe that in any case $h^{0,1} = h^{1,0}$ follows by reduction to the Picard variety of X (this needs torsion-freeness of $H^2_{\mathrm{cris}}(X/W)$ which enables us to deduce that $\mathrm{Pic}(X)$ is reduced (see [9, II, 5.16]); also see Remark 2) , where the assertion is trivial and this proves the result for $\dim(X) = 2$. So we are left with $\dim(X) = 3$ and have to prove that $h^{0,2} = h^{2,0}$.

One first notes that $H^*(X, W\Omega^\bullet_X)$ is a Mazur–Ogus object in the derived category of bounded complexes of modules over the Cartier–Dieudonne–Raynaud algebra. To see this one observes that the slope spectral sequence computes the crystalline cohomology of X (see [9, 3.1.1, p. 614]) and that $H^*(X, W\Omega^\bullet_X)$ is a coherent module over the Cartier–Dieudonne–Raynaud algebra (see [11, II, Theorem 2.2]). By our hypothesis the crystalline cohomology of X is torsion free, so by the universal coefficient theorem we see that:

$$\mathrm{rank}_W H^n_{\mathrm{cris}}(X/W) = \dim_k H^n_{\mathrm{dR}}(X/k).$$

Finally as the Hodge–de Rham spectral sequence of X degenerates at E_1, we see that the number on the right is equal to $\sum_{i+j=n} \dim H^i(X, \Omega^j_X)$. Hence by the definition of Mazur–Ogus objects (see [5, IV, 1.1]) we see that $H^*(X, W\Omega^\bullet_X)$ is a Mazur–Ogus object. Hence we can apply [5, Corollary 3.3.1, p. 86] to see that $h^{i,j}_W = h^{i,j}$ and so by Lemma 2.2 we are done. □

Proof of Theorem 3.1. Now to prove Theorem 3.1 it will suffice to reduce to the situation where we can apply Theorem 3.2. This is done by a "spreading out" argument as in [2, Corollary 2.4]. Let n be the dimension of $X/\mathbb{C}$. By writing $\mathbb{C}$ as a direct limit of its subrings of finite type over $\mathbb{Z}$, we can assume that X arises by extension of scalars from a projective and smooth scheme of finite type $\mathcal{X}/S$, which is pure of relative dimension n, and where S is affine, integral, smooth of finite type over $\mathbb{Z}$, and the relative Hodge and de Rham cohomology groups of $\mathcal{X}/S$ are locally free of finite type and so commute with base extensions. Further shrinking S if required we can also assume that the Hodge to de Rham spectral sequence of $\mathcal{X}/S$ degenerates at E_1, and that $n!$ is invertible in $\mathcal{O}_S$.

Now to complete the proof we choose a closed point of S and apply Theorem 3.2 to the fibre over it, and note that the hypothesis of the theorem are satisfied. Indeed the degeneration of Hodge to de Rham spectral sequence on the special fibre follows from Deligne–Illusie criterion for degeneration (see [2, Corollary 2.4]) and the torsion-freeness of crystalline cohomology of the special fibre follows from the comparison theorem between de Rham and crystalline cohomologies and the fact that the relative de Rham cohomology of $\mathcal{X}/S$ is locally free. □

3.1. Some remarks

Remark 2. The assumption that the crystalline cohomology of X is torsion free is a necessary assumption in Theorem 3.2. If this assumption is dropped then Hodge symmetry fails in positive characteristic. Here is one example (for surfaces): this is taken from [9, Chapter II, Section 7.3]. Let X be a smooth projective Enriques surface in characteristic two. Assume that X is singular, i.e., $H^1(X, \mathcal{O}_X)$ is one-dimensional and Frobenius is bijective on this vector space; such surfaces exist only in characteristic two. Then one has a complete list of the Hodge invariants of X. In particular in the present situation, there are no global one forms on X but as $H^1(X, \mathcal{O}_X) \neq 0$, the Picard scheme of X is not reduced (it is equal to μ_2 so the Albanese variety is zero and hence the second crystalline cohomology of X has torsion (it is of V-torsion type in Illusie's classification of torsion). In fact as the proof of [9, Proposition 7.3.5, p. 656] shows, the cohomology of X with coefficients in the sheaf of Witt vectors is of finite type and so X is Hodge–Witt. Other examples of surfaces (in any characteristic p) where Hodge symmetry fails can be found in [15].

One would like to conjecture that Hodge–Witt symmetry should hold under some reasonable hypothesis. Here is a precise statement:

Conjecture 1. Suppose that X/k is a smooth, projective variety over a perfect field. Assume that X lifts to W_2 and the crystalline cohomology of X is torsion free. Then X satisfies Hodge–Witt symmetry. In particular

Ekedahl's conditions hold for X, that is for all i, j, the dimension, $T^{i,j}$ of the domino associated to the differential $d : H^j(X, W\Omega^i) \to H^j(X, W\Omega^{j+1})$ satisfies

$$T^{i,j} = T^{j-2,i+2}. \tag{11}$$

Remark 3. We recall that the equality (11) was shown by Ekedahl to be a necessary and sufficient condition for Hodge–Witt symmetry to hold (see [5, VI, 3.2 (ii)]). We refer to this equality as "Ekedahl's condition for Hodge–Witt symmetry". We note that the proof of Theorems 3.1, 3.2 can be turned around: as one does know that Hodge symmetry does hold over complex numbers, one can use the proof just given to deduce Hodge–Witt symmetry holds on the special fibre. Here is a precise assertion.

Proposition 3.1. *Let X/k be a smooth projective variety over a perfect field of characteristic $p > 0$. Assume that X admits a smooth, projective lifting to Witt vectors $W(k)$ of k, and that $p > \dim(X)$. Assume further that the crystalline cohomology of X is torsion free. Then Hodge–Witt symmetry holds for X.*

Proof. Choose a smooth lifting Y/W of X. We deduce the fact that $H^*(X, W\Omega_X^\bullet)$ is a Mazur–Ogus object exactly as in the proof of Theorem 3.2. This implies that the Hodge numbers of X are the same as the Hodge–Witt numbers of X. By the torsion freeness of the crystalline cohomology we deduce that the de Rham cohomology of Y/W is torsion free as well. Now by [6], the Hodge de Rham spectral sequence of Y/W degenerates at E_1 and the Hodge filtration is by direct summands, hence the all the Hodge groups $H^j(Y, \Omega^i_{Y/W})$ are torsion free as well. By extending scalars from W to $\mathbb{C}$, and using the Hodge theorem we deduce that Hodge–Witt symmetry holds. By [5, IV, 3.2 (ii)] we see that (11) is satisfied. $\square$

Remark 4. In a forthcoming work (see [13]) we have investigated properties of Hodge–Witt numbers and slope numbers in detail. For instance we have shown that for all smooth projective surfaces of general type which lift to W_2 and have torsion free crystalline cohomology, the following inequality holds

$$c_1^2 \leqslant 5c_2 + 6b_1. \tag{12}$$

Here c_1, c_2 have their usual meaning and b_1 is the first Betti number of X computed, say, using étale cohomology. For this and other investigations on Hodge–Witt numbers we refer the reader to [13].

Remark 5. In general the conditions of Theorem 3.2 are not very easy to verify. But here is one application: assume that X is Frobenius split and that $p \geqslant 5$. Then by a result of Mehta (see [12]) we see that Hodge de Rham spectral sequence of X degenerates at E_1, so if crystalline cohomology of X is torsion free then X satisfies Hodge symmetry.

Acknowledgements

We would like to thank Luc Illusie for correspondence and comments; we would also like to thank Minhyong Kim for discussions on Hodge symmetry and also to Jim Carlson whose lectures on mixed Hodge theory at the University of Arizona revived our interest in algebraic proofs of Hodge symmetry. We are grateful to F. Oort for his interest and his suggestions which have improved the readability of this paper. We also thank Dinesh Thakur for comments. We thank the referee for a number of suggestions which have improved the readability of this manuscript.

References

[1] P. Deligne, La conjecture de Weil II, Publ. Math. I.H.E.S. 52 (1980) 137–252.
[2] P. Deligne, L. Illusie, Relèvements modulo p^2 et décomposition du complexe de de Rham, Invent. Math. 89 (2) (1987) 247–270.

[3] T. Ekedahl, On the multiplicative properties of the de Rham–Witt complex I, Ark. Mat. 22 (1984) 185–239.
[4] T. Ekedahl, On the multiplicative properties of the de Rham–Witt complex II, Ark. Mat. 23 (1985).
[5] T. Ekedahl, Diagonal Complexes and F-Gauge Structures, in: Travaux en Cours, Hermann, Paris, 1986.
[6] J.-M. Fontaine, W. Messing, p-adic periods and p-adic étale cohomology, in: Current Trends in Arithmetical Algebraic Geometry, (Arcata), in: Contemp. Math., vol. 67, 1987, pp. 179–207.
[7] P. Griffiths, Principles of Algebraic Geometry, reprint of the 1978 edition, Wiley Classics Lib., Wiley, New York, 1994.
[8] R. Hartshorne, Algebraic Geometry, in: Graduate Texts in Math., vol. 52, Springer-Verlag, New York, 1977.
[9] L. Illusie, Complexe de de Rham–Witt et cohomologie cristalline, Ann. Sci. École Norm. Sup. 12 (1979) 501–661.
[10] L. Illusie, Finiteness, duality and Künneth theorems in the cohomology of the de Rham–Witt complex, in: Algebraic Geometry Tokyo/Kyoto, in: Lecture Notes in Math., vol. 1016, Springer-Verlag, 1983, pp. 20–72.
[11] L. Illusie, M. Raynaud, Les suites spectrales associées au complexe de de Rham–Witt, Publ. Math. I.H.E.S. 57 (1983) 73–212.
[12] K. Joshi, Exotic torsion in Frobenius split varieties, Preprint, 2000.
[13] K. Joshi, Crystalline aspects of geography of low dimensional varieties, in preparation.
[14] N. Katz, W. Messing, Some consequences of the Riemann hypothesis for varieties over finite fields, Invent. Math. 23 (1974) 73–77.
[15] J.-P. Serre, Sur la topologie des variétés algébriques en caractéristique p, Symp. Topology, Mexico (1956) 24–53.