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Eigenvalue asymptotics of a modified Jaynes–Cummings model with periodic modulations

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Abstract

We analyze the influence of additive and multiplicative periodic modulations on the asymptotic behavior of eigenvalues of some Hermitian Jacobi Matrices related to the Jaynes–Cummings model using the so-called "successive diagonalization" method. This approach allows us to find the asymptotics of the nth eigenvalue λ_n as $n \to \infty$ stepwise with successively increasing precision. We bring to light the interplay of additive and multiplicative periodic modulations and their influence on the asymptotic behavior of eigenvalues. ***To cite this article: A. Boutet de Monvel et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Effet de modulations périodiques sur l'asymptotique des valeurs propres d'une variante du modèle de Jaynes–Cummings. L'objet de cette Note est d'analyser l'effet de modulations périodiques additives et multiplicatives sur le comportement asymptotique des valeurs propres de matrices de Jacobi liées au modèle de Jaynes–Cummings. Nous utilisons une méthode « de diagonalisations successives » pour obtenir le comportement asymptotique, pour $n \to +\infty$, de la $n^{\text{ième}}$ valeur propre λ_n, celles-ci étant supposées rangées par ordre croissant. Les résultats obtenus mettent en évidence l'effet des modulations périodiques considérées sur le comportement asymptotique des valeurs propres. ***Pour citer cet article : A. Boutet de Monvel et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

Soit $\mathbb{N} = \{1, 2, \ldots\}$ l'ensemble des entiers $n > 0$. On note $l^2(\mathbb{N})$ l'espace de Hilbert des suites complexes $u = \{u_n\}_{n=1}^{\infty}$ de carré sommable et $l_{\text{fini}}(\mathbb{N})$ le sous-espace des u telles que $u_n = 0$ pour tout n assez grand.

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Soient $b = \{b_n\}_{n\in\mathbb{N}}$ et $q = \{q_n\}_{n\in\mathbb{N}}$ deux suites réelles, $b_n > 0$. Soit $J = J_{b,q}$ l'opérateur dans $l^2(\mathbb{N})$ défini pour tout $u \in l_{\text{fini}}(\mathbb{N})$ par

$$(Ju)_1 = q_1u_1 + b_1u_2, \tag{1}$$

$$(Ju)_n = b_{n-1}u_{n-1} + q_nu_n + b_nu_{n+1}, \quad n \geqslant 2. \tag{2}$$

Cet opérateur a pour matrice dans la base hilbertienne standard de $l^2(\mathbb{N})$ la « matrice de Jacobi »

$$\begin{pmatrix} q_1 & b_1 & 0 & 0 & \cdots \\ b_1 & q_2 & b_2 & 0 & \cdots \\ 0 & b_2 & q_3 & b_3 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{3}$$

Dans cette Note on s'intéresse au cas où les suites b et q sont de la forme

$$b_n = \mathcal{B}_n\sqrt{n} \quad \text{et} \quad q_n = n^2 + \mathcal{E}_n, \tag{4}$$

où $\{\mathcal{B}_n\}_{n\in\mathbb{N}}$ et $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ sont deux suites réelles périodiques de période $T \in \mathbb{N}$:

$$\mathcal{B}_n = \mathcal{B}_{n+T} > 0 \quad \text{et} \quad \mathcal{E}_n = \mathcal{E}_{n+T}. \tag{5}$$

Ce type de matrice de Jacobi est inspiré du modèle « de Jaynes–Cummings », qui est le modèle le plus simple d'interaction électromagnétique entre un atome à deux niveaux et un champ unimodal. On peut en effet montrer (voir [7]) que l'étude des propriétés spectrales du Hamiltonien de Jaynes–Cummings (6) se ramène à l'analyse spectrale d'une matrice de Jacobi du type (3), pour des suites b et q de la forme

$$b_n = g\sqrt{n} \quad \text{et} \quad q_n = n\omega + E_n, \quad E_n = E_{n+2}, \tag{6}$$

où $\{E_n\}$ est une suite réelle 2-périodique, et où g et ω sont deux constantes > 0.

L'objectif de cette Note est d'analyser l'effet des modulations périodiques $\{\mathcal{B}_n\}$ et $\{\mathcal{E}_n\}$ sur le comportement asymptotique, pour $n \to +\infty$, de la $n^{\text{ième}}$ valeur propre λ_n de l'opérateur $J = J_{b,q}$. Par rapport au modèle de Jaynes–Cummings, nous avons modifié, par commodité, la partie non modulée de la diagonale, de $n\omega$ en n^2, et nous avons considéré une modulation additive $\mathcal{E}_n$ de période T au lieu de 2, par référence à un atome à T niveaux. Quant à l'introduction d'une modulation multiplicative $\{\mathcal{B}_n\}$ à la place de la constante g, elle correspond à une formulation plus générale du modèle de Jaynes–Cummings [7].

Les deux théorèmes de cette Note donnent le comportement asymptotique de la $n^{\text{ième}}$ valeur propre λ_n de J, les λ_n étant supposées rangées par ordre croissant :

(i) en l'absence de modulations, i.e., $T = 1$:

$$\lambda_n = n^2 + \mathcal{E}_1 + \frac{\mathcal{B}_1^3 - 4\mathcal{B}_1}{16n^2} + \mathrm{O}\left(\frac{1}{n^2\sqrt{n}}\right),$$

(ii) en présence de modulations, i.e., $T > 1$:

$$\lambda_n = n^2 + \mathcal{E}_n + \frac{\mathcal{B}_{n-1}^2 - \mathcal{B}_n^2}{2} + \mathrm{O}\left(\frac{1}{\sqrt{n}}\right).$$

Les démonstrations utilisent systématiquement une méthode « de diagonalisations successives » déjà utilisée dans [2] et inspirée de Rozenblum [6] et de la méthode de l'opérateur de transformation. Dans une première étape, on obtient une matrice $\tilde{J}_1$ dont on sait déterminer très facilement le comportement asymptotique de la $n^{\text{ième}}$ valeur propre $\lambda_n(\tilde{J}_1)$, par exemple une matrice diagonale, et qui est « presque semblable » à J, ce qui permet d'estimer la différence $\lambda_n(J) - \lambda_n(\tilde{J}_1)$ pour $n \to +\infty$. À l'étape suivante, on obtient une matrice $\tilde{J}_2$ qui a des propriétés analogues à celles de $\tilde{J}_1$, mais dont les valeurs propres donnent une meilleure approximation asymptotique des valeurs propres de J pour $n \to +\infty$. Et ainsi de suite. Les preuves détaillées sont données dans [1].

1. Introduction

Let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of positive integers. Let $l^2(\mathbb{N})$ be the Hilbert space of square sommable complex sequences $u = \{u_n\}_{n=1}^{\infty}$ and $l_{\text{fini}}(\mathbb{N})$ the subspace of u such that $u_n = 0$ for n large enough.

Let $b = \{b_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers, and $q = \{q_n\}_{n\in\mathbb{N}}$ a sequence of real numbers. In $l^2(\mathbb{N}) = l^2(\mathbb{N}, \mathbb{C})$ let $J = J_{b,q}$ be the operator defined for every $u \in l_{\text{fini}}(\mathbb{N})$ by

$$(Ju)_1 = q_1 u_1 + b_1 u_2, \tag{1}$$

$$(Ju)_n = b_{n-1}u_{n-1} + q_n u_n + b_n u_{n+1}, \quad n \geqslant 2. \tag{2}$$

This operator is symmetric, therefore closable, and whenever we refer to this operator's spectrum we shall have in mind its closure's spectrum. The matrix representation of this operator with respect to the canonical basis in $l^2(\mathbb{N})$ is the Hermitian Jacobi matrix

$$\begin{pmatrix} q_1 & b_1 & 0 & 0 & \cdots \\ b_1 & q_2 & b_2 & 0 & \cdots \\ 0 & b_2 & q_3 & b_3 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{3}$$

In this Note we assume $b = \{b_n\}_{n\in\mathbb{N}}$ and $q = \{q_n\}_{n\in\mathbb{N}}$ of the form

$$b_n = \mathcal{B}_n\sqrt{n} \quad \text{and} \quad q_n = n^2 + \mathcal{E}_n \quad \text{for all } n \in \mathbb{N}, \tag{4}$$

where $\{\mathcal{B}_n\}_{n\in\mathbb{N}}$ and $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ are real periodic sequences of period $T \in \mathbb{N}$, with $\{\mathcal{B}_n\}$ positive:

$$\mathcal{B}_n = \mathcal{B}_{n+T} > 0 \quad \text{and} \quad \mathcal{E}_n = \mathcal{E}_{n+T} \quad \text{for all } n \in \mathbb{N}. \tag{5}$$

It follows straightforwardly from (4) that J is essentially self-adjoint and has discrete spectrum. Indeed, J is a relatively compact perturbation of a diagonal operator [3].

The main goal here is to find the exact asymptotics of the eigenvalues of J using the so-called method of "successive diagonalization". In [2], this approach has already been used successfully for obtaining the exact asymptotic behavior of the spectrum of a class of Jacobi matrices which, as in our case, may be of physical interest. The idea behind the method of "successive diagonalization" goes back to the method of transformation operator, but using a successive approximation approach. Rozenblum developed somewhat similar ideas for the case of pseudo-differential operators on the circle [6]. The point is that under certain conditions on J we can find a diagonal matrix,[1] say $\tilde{J}_1$, which is similar to the original matrix J modulo some compact matrix with special structure. Then, it turns out that it is possible to estimate, asymptotically as $n \to \infty$, the distance between the eigenvalues of J and the eigenvalues of the obtained matrix $\tilde{J}_1$. Moreover, the procedure for obtaining $\tilde{J}_1$ allows us to obtain a subsequent matrix $\tilde{J}_2$ which is similar to the matrix J modulo some other compact matrix such that the asymptotic behavior of J's eigenvalues is expressed by means of the eigenvalues of $\tilde{J}_2$ with greater precision than in the previous step using $\tilde{J}_1$. So, it is possible to apply this procedure repeatedly and thus, to obtain the asymptotic behavior of J's eigenvalues as $n \to \infty$ with arbitrary precision.

Our choice of the sequences b and q in Eq. (4) is inspired on the Jaynes–Cummings model, which is the simplest model for describing the electromagnetic interaction between a two-level atom and a single mode field. In this model the Hamiltonian is given by

$$\boldsymbol{H} = \omega_0\sigma_0 + \omega\boldsymbol{a}^+\boldsymbol{a} + g\sigma_1(\boldsymbol{a} + \boldsymbol{a}^+), \tag{6}$$

where ω_0 is the atomic transition frequency, ω is the field mode frequency, $\boldsymbol{a}$ and $\boldsymbol{a}^+$ are the photon creation and annihilation operators, g is the coupling constant, and $\sigma_0 = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. In this model the counter-rotating

[1] Or at least a matrix such that the asymptotic behavior of its eigenvalues can easily be found.

term is considered [4,5]. Despite the seeming simplicity, this model cannot be solved exactly. Nevertheless it has been found that the study of the spectral properties of (6) can be reduced to the spectral analysis of an Hermitian Jacobi matrix (see [7]) of the form (3) with $b = \{b_n\}_{n\in\mathbb{N}}$ and $q = \{q_n\}_{n\in\mathbb{N}}$ given by

$$b_n = g\sqrt{n} \quad \text{and} \quad q_n = n\omega + E_n, \tag{7}$$

where $\{E_n\}_{n\in\mathbb{N}}$ is a real 2-periodic sequence ($T = 2$), and ω and g two positive constants.

Since we are mostly interested in how the modulations affect the asymptotic behavior of eigenvalues, we have modified the Jacobi matrix obtained from the Jaynes–Cummings model stressing the role of the modulations and in such a way that the calculations involved in the method of successive diagonalization are easily carried out (compare (4) and (7)). If we had not changed the growing rate of the main diagonal, then the additive modulation $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ proposed in our model would correspond to a T-level atom. On the other hand, the introduced multiplicative modulation, $\{\mathcal{B}_n\}_{n\in\mathbb{N}}$, can arise when considering a more general setting of the Jaynes–Cummings model [7].

The successive diagonalization approach used in this work allows us to disclose the asymptotics of eigenvalues in the case of additive and multiplicative periodic modulations and to study the interplay of the modulations and their influence on the asymptotic behavior of eigenvalues.

2. Asymptotics of eigenvalues for the case without modulations

The case without modulations, i.e., $T = 1$, is rather elementary, but we consider it separately as a reference for comparison to the case with modulated entries. The following result shows that when the sequences $\{\mathcal{B}_n\}_{n=1}^{\infty}$ and $\{\mathcal{E}_n\}_{n=1}^{\infty}$ are constants, i.e., $\mathcal{B}_n = \mathcal{B}_1$ and $\mathcal{E}_n = \mathcal{E}_1$ for all $n \in \mathbb{N}$, then $\mathcal{B}_1$ firstly appears in the term $\frac{1}{n^2}$ of the asymptotic expansion of the n^{th} eigenvalue λ_n. We proved this assertion applying straightforwardly the successive diagonalization method in its simple form.

Theorem 2.1 (without modulations). *Let $J = J_{b,q}$ be the operator* (1), (2) *with b, q given by* (4), (5), *and $T = 1$. Let $\sigma(J) = \{\lambda_n\}_{n=1}^{\infty}$ be the spectrum of J, the λ_n's being enumerated in ascending order. Then λ_n behaves asymptotically as $n \to \infty$ as follows*

$$\lambda_n = n^2 + \mathcal{E}_1 + \frac{\mathcal{B}_1^3 - 4\mathcal{B}_1}{16n^2} + \mathrm{O}\left(\frac{1}{n^{5/2}}\right). \tag{8}$$

3. Asymptotics of eigenvalues for the case of additive and multiplicative periodic modulations

Let us assume that the modulation sequences $\{\mathcal{B}_n\}_{n\in\mathbb{N}}$ and $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ have a period $T > 1$. In this case we have to consider the Jacobi matrix (3) as a block matrix (see below). It turns out that, when applying the successive diagonalization technique to generalized Jacobi matrices (block matrices), matrix equations of commutator type arise. This is the main difference between the application of the successive diagonalization method in the previous section and the present one and, to some extent, the equations involved in the proof of this section's main result resemble the multidimensional case.

We introduce the $T \times T$ matrices

$$B_n = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ b_{nT} & 0 & \dots & 0 \end{pmatrix}$$

and

$$Q_n = \begin{pmatrix} q_{nT-(T-1)} & b_{nT-(T-1)} & & & & & \\ b_{nT-(T-1)} & q_{nT-(T-2)} & b_{nT-(T-2)} & & & & \\ & b_{nT-(T-2)} & q_{nT-(T-3)} & b_{nT-(T-3)} & & & \\ & \ddots & \ddots & \ddots & & & \\ & & b_{nT-3} & q_{nT-2} & b_{nT-2} & \\ & & & b_{nT-2} & q_{nT-1} & b_{nT-1} \\ & & & & b_{nT-1} & q_{nT} \end{pmatrix}.$$

Consider the following generalized Jacobi matrix (block matrix):

$$\begin{pmatrix} Q_1 & B_1 & & & \\ B_1^* & Q_2 & B_2 & & \\ & B_2^* & Q_3 & B_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{9}$$

This matrix can be seen as the matrix representation of the operator $\mathcal{J}$ in $l^2(\mathbb{N}, \mathbb{C}^T)$ defined for every sequence $w = \{w_n\}_{n\in\mathbb{N}} \in l_{\mathrm{fini}}(\mathbb{N}, \mathbb{C}^T)$ by

$$(\mathcal{J}w)_1 = Q_1 w_1 + B_1 w_2, \tag{10}$$

$$(\mathcal{J}w)_n = B_{n-1} w_{n-1} + Q_n w_n + B_n w_{n+1}, \quad n \geqslant 2. \tag{11}$$

It is clear that $\mathcal{J}$ is unitarily equivalent to the original operator J. Thus, we study the asymptotic behavior of the spectrum of J by means of $\mathcal{J}$ using the successive diagonalization approach. Our result is the following

Theorem 3.1 (with modulations). *Let $J = J_{b,q}$ be the operator* (1), (2) *with b, q given by* (4), (5), *and $T > 1$. Let $\sigma(J) = \{\lambda_n\}_{n=1}^{\infty}$ be the spectrum of J, the λ_n's being enumerated in ascending order. Then λ_n behaves asymptotically as $n \to \infty$ as follows*

$$\lambda_n = n^2 + \mathcal{E}_n + \frac{\mathcal{B}_{n-1}^2 - \mathcal{B}_n^2}{2} + \mathrm{O}\left(\frac{1}{\sqrt{n}}\right). \tag{12}$$

Thus, the effect of the sequence $\{\mathcal{B}_n\}_{n=1}^{\infty}$ on the asymptotic behavior of eigenvalues is sharply increased ($\mathrm{O}(1/n^2) \mapsto \mathrm{O}(1)$) when $\{\mathcal{B}_n\}_{n=1}^{\infty}$ is periodically modulated, i.e., when $T > 1$. Notice that in this case an appropriate choice of the modulations may cancel the $\mathrm{O}(1)$ term of the asymptotic expansion.

The detailed proofs of both theorems are in [1].

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