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Differential Geometry

A Note on pinching sphere theorem

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Abstract

Let M^{2n} be a $2n$-dimensional compact, simply connected Riemannian manifold without boundary and S^{2n} be the unit sphere of $2n+1$ dimension Euclidean space $\mathbb{R}^{2n+1}$. We prove in this note that if the sectional curvature K_M varies in $(0,1]$ and the volume $V(M)$ is not larger than $(\frac{3}{2}+\eta)V(S^{2n})$ for some positive number η depending only on n, then M^{2n} is homeomorphic to S^{2n}. ***To cite this article: Y. Wen, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Une Note sur le théorème de la sphère pincée. Soit M^{2n} une variété riemannienne compacte, simplement connexe de dimension $2n$ sans bord et soit S^{2n} la sphère unitée de l'espace euclidien $\mathbb{R}^{2n+1}$. Nous prouvons que si la courbure sectionnelle K_M varie dans $]0,1]$ et si le volume $V(M)$ est inférieur à $(\frac{3}{2}+\eta)V(S^{2n})$ pour un nombre positif η dependant seulement de n, alors M^{2n} est homéomorphe à S^{2n}. ***Pour citer cet article : Y. Wen, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

En géométrie différentielle, les théorèmes de la sphère pincée ont toujours été un sujet intéressant. Un des resultats célèbres est le théorème de la sphère $\frac{1}{4}$-pincée affirmant qu'une variété riemannienne M^n compacte, simplement connexe, à courbure sectionnelle $K_M \in]\frac{1}{4},1]$ est homéomorphe à la sphère unité S^n de l'espace euclidien $\mathbb{R}^{n+1}$. Après cela, d'autres conditions géometriques ou topologiques ont été considérées (voir, par exemple, [2,3,5–7]). En 1991, Coghlan et Itokawa ont conjecturé dans [2] que pour toute variété riemannienne M^{2n} compacte, simplement connexe de dimension $2n$, si on suppose que $0 < K_M \leqslant 1,\ 0 < V(M) < 3V(S^{2n})$, où $V(M)$ désigne le volume de M^{2n}, alors M^{2n} est homéomorphe à S^{2n}. Ils ont démonté le résultat lorsque $0 < V(M) \leqslant \frac{3}{2}V(S^{2n})$. Ils ont aussi exhibé un contre exemple dans le cas $V(M) = 3V(S^{2n})$ (voir [2]). Il semble que peu de progrès a été réalisé depuis sur ce problème. Dans cette Note, nous démontrons tout d'abord :

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Théorème 0.1. *Soit M^{2n} une variété riemannienne compacte, simplement connexe de dimension $2n$. On suppose que $0 < K_M \leqslant 1$ et que M^{2n} n'est pas homéomorphe à S^{2n}. Alors il existe trois points x_0, p et q dans M^{2n} tels que*

$$d(p, x_0) \geqslant \pi, \quad d(q, x_0) \geqslant \pi, \quad \text{et} \quad d(p, q) = d_M, \tag{$*$}$$

où $d(x, y)$ exprime la distance entre les points x et y associée à g et d_M est le diamètre de M^{2n}. De plus, on a $V(M) \geqslant \frac{3}{2} V(S^{2n})$.

Une conséquence immédiate du théorème 0.1 (voir [2]) est le :

Théorème 0.2. *Soit M^{2n} une variété riemannienne compacte, simplement connexe de dimension $2n$. Si $0 < K_M \leqslant 1$ et $0 < V(M) < \frac{3}{2} V(S^{2n})$, M^{2n} est homéomorphe à S^{2n}.*

On démontre aussi l'extension suivante :

Théorème 0.3. *Pour $n \geqslant 1$, il existe $\eta > 0$, dépendant seulement de n, tel que si (M^{2n}, g) est une variété riemannienne compacte, simplement connexe de dimension $2n$ qui satisfait*

$$0 < K_M \leqslant 1 \quad \text{et} \quad \frac{3}{2} V\big(S^{2n}\big) \leqslant V(M) \leqslant \left(\frac{3}{2} + \eta\right) V\big(S^{2n}\big), \tag{$**$}$$

alors M^{2n} est homéomorphe à S^{2n}.

Un résultat similaire a été annoncé dans [6], mais la preuve présentée ne semblait pas être convaincante.

Pour dḿontrer le Théorème 0.1, nous remarquons simplement que si M^{2n} est la réunion de deux boules de rayon strictement inférieur à π, i.e. $M^{2n} = \mathbb{B}_{\pi-\varepsilon}(p) \cup \mathbb{B}_{\pi-\varepsilon}(q)$ avec $\varepsilon > 0$, alors un résultat classique dit que M^{2n} est homéomorphe à S^{2n}. Cela nous permet de prouver l'existence des points p, q et x_0 satisfaisant $(*)$. Le Théorème du Volume Comparatif donne ensuite $V(M) \geqslant \frac{3}{2} V(S^{2n})$.

Pour démontrer Théorème 0.3, nous procédons par l'absurde. Nous supposons qu'il existe une suite de variété riemannienne $\{M_i^{2n}, g_i\}$ telle que $\lim_{i\to\infty} V(M_i) = \frac{3}{2} V(S^{2n})$ et M_i^{2n} n'est pas homéomorphe à S^{2n}. Nous montrons alors que $\lim_{i\to\infty} d_i = \pi$, où d_i désigne le diamètre de M_i^{2n}. Et puis, en analysant la suite des points (p_i, q_i, x_i) donnés par Théorème 0.1, nous aboutissons à une contradiction en utilisant le Théorème du Volume Compartif et le Théorème de Toponogov.

1. Introduction

In differential geometry, the sphere theorem has been always an interesting subject. One of the most known results is the so called $\frac{1}{4}$-pinching sphere theorem, which states that every n-dimensional simply connected, compact, without boundary Riemannian manifold M^n with the sectional curvature K_M in $(\frac{1}{4}, 1]$ is homeomorphic to the n-dimensional standard unit sphere S^n of $\mathbb{R}^{n+1}$ (see [1,5]). Later, some different geometric or topological restrictions were also considered (see, for instance, [2,8,6,7,4]). In 1991, Coghlan and Itokawa conjectured in [2] that a $2n$-dimensional simply connected, compact, without boundary Riemannian manifold M^{2n}, satisfying $0 < K_M \leqslant 1$ and $0 < V(M) < 3V(S^{2n})$, would be homeomorphic to S^{2n}, where $V(M)$ denotes the volume of M^{2n}. In the same paper, they proved that it is true under the assumptions $0 < K_M \leqslant 1$ and $0 < V(M) \leqslant \frac{3}{2} V(S^{2n})$ and gave a counterexample which indicated that the volume condition $3V(S^{2n})$ is sharp. As far as we know, it seems that no further progress has been made on this problem. In this Note, we obtain a pinching sphere theorem with a little bit more relaxed condition than that in [2] (see Theorem 2.2).

2. Results

We denote by d_M the diameter of M^{2n}, i_M the injectivity radius, and $\gamma_{x,y}$ a minimal geodesic connecting two points x and y in M^{2n}. We note also $d(x,y)$ the distance between x and y with respect to the metric g. $\mathbb{B}_r(x)$ denotes the open geodesic ball in M^{2n} centered at x with radius r, $\partial\mathbb{B}_r(x)$ the boundary of $\mathbb{B}_r(x)$ and $\overline{\mathbb{B}}_r(x)$ the closure of $\mathbb{B}_r(x)$. Now we first prove the following

Theorem 2.1. *Suppose that $0 < K_M \leqslant 1$ and M^{2n} is not homeomorphic to S^{2n}, then there exist three points x_0, p and q in M^{2n} such that $d(p,x_0) \geqslant \pi$, $d(q,x_0) \geqslant \pi$ and $d_M = d(p,q)$. In addition, we have $V(M) \geqslant \frac{3}{2}V(S^{2n})$.*

Remark 1. For any even dimensional simply connected, compact, without boundary Riemannian manifold M satisfying $0 < K_M \leqslant 1$, we have that the injectivity radius $i_M \geqslant \pi$ (see [5], p. 198, Corollary 1.8).

Proof of Theorem 2.1. By Remark 1, $i_M \geqslant \pi$, thus $d_M \geqslant \pi$. Let $p, q \in M^{2n}$ such that $d_M = d(p,q)$. If there exists a positive number ε which satisfies

$$\overline{\mathbb{B}}_{\pi-\varepsilon}(p) \cup \overline{\mathbb{B}}_{\pi-\varepsilon}(q) = M^{2n},$$

according to the classical topology theorem (see [1,2]), M^{2n} will be homeomorphic to S^{2n}. Therefore, for each $m \in \mathbb{N}^*$, there exists $x_m \in M^{2n}$ such that

$$d(p,x_m) > \pi - \frac{1}{m}, \qquad d(q,x_m) > \pi - \frac{1}{m}.$$

We can thus choose a subsequence of $\{x_m\}$ which converges to a point x_0 of M^{2n}. Hence $d(p,x_0) \geqslant \pi$ and $d(q,x_0) \geqslant \pi$. We consider then the pairwise disjoint geodesic balls $\mathbb{B}_{\pi/2}(p)$, $\mathbb{B}_{\pi/2}(q)$ and $\mathbb{B}_{\pi/2}(x_0)$. Using the Günther–Bishop Volume Comparison Theorem (V.C.T.) (see [3], p. 144, Theorem 3.101), it comes that $V(M) \geqslant \frac{3}{2}V(S^{2n})$, since $0 < K_M \leqslant 1$. □

As a consequence (see [2]), we immediately have:

Theorem 2.2. *Let (M^{2n}, g) is a $2n$-dimensional compact, simply connected, Riemannian manifold without boundary, with the metric g. If M^{2n} satisfies*

$$0 < K_M \leqslant 1, \qquad 0 < V(M) < \frac{3}{2}V\big(S^{2n}\big),$$

then M^{2n} is homeomorphic to S^{2n}.

Now we state our main result as following:

Theorem 2.3. *For any $n \in \mathbb{N}^*$, there exists a positive number η depending only on n such that if (M^{2n}, g) is a $2n$-dimensional compact, simply connected, Riemannian manifold without boundary, with the metric g and satisfies*

$$0 < K_M \leqslant 1, \qquad \frac{3}{2}V\big(S^{2n}\big) \leqslant V(M) \leqslant \left(\frac{3}{2} + \eta\right)V\big(S^{2n}\big),$$

then M^{2n} is homeomorphic to S^{2n}.

Remark 2. A similar but stronger result was announced in [7]. However, the proof presented there seems to have a gap.

3. Proof of Theorem 2.3

We prove this theorem by contradiction. If not, there exists a decreasing positive sequence $\{\eta_i\}$ satisfying that $\eta_i \to 0$ as $i \to \infty$ and such that a sequence of $2n$-dimensional Riemannian manifolds $\{(M_i, g_i)\}$ exists which satisfies, for each i,

$$0 < K_{M_i} \leqslant 1, \qquad \frac{3}{2}V\left(S^{2n}\right) \leqslant V(M_i) \leqslant \left(\frac{3}{2}+\eta_i\right)V\left(S^{2n}\right),$$

and M_i is not homeomorphic to S^{2n}. First, we show that

$$d_i \to \pi \quad \text{as } i \to \infty, \tag{1}$$

where d_i is the diameter of M_i with respect to the metric g_i. For that, let $p_i,\ q_i$ be some points of M_i with $d(p_i, q_i) = d_i$, where $d(p_i, q_i)$ is the distance between points p_i and q_i with respect to g_i of M_i. By Theorem 2.1, for each i, there exists $x_i \in M_i$ which satisfies $d(p_i, x_i) \geqslant \pi$ and $d(q_i, x_i) \geqslant \pi$. Due to V.C.T., it is clear that $d_i < 2\pi$ for each i. It is also clear that the three geodesic balls $\mathbb{B}_{d_i/2}(p_i), \mathbb{B}_{d_i/2}(q_i)$ and $\mathbb{B}_{\pi-d_i/2}(x_i)$ in M_i are pairwise disjointed. Using once again V.C.T., we can see that

$$\begin{aligned}\left(\frac{3}{2}+\eta_i\right)V\left(S^{2n}\right) &\geqslant V(M_i) \geqslant V\left(\mathbb{B}_{d_i/2}(p_i)\right) + V\left(\mathbb{B}_{d_i/2}(q_i)\right) + V\left(\mathbb{B}_{\pi-d_i/2}(x_i)\right)\\ &\geqslant V\left(S^{2n}\right) + V\left(\mathbb{B}^s_{d_i/2}\right) \geqslant \frac{3}{2}V\left(S^{2n}\right),\end{aligned}$$

where $\mathbb{B}^s_\delta$ is the geodesic ball on S^{2n} with radius δ. As $i \to \infty,\ \eta_i \to 0$, thus $d_i \to \pi$. We set that for each i,

$$r_i = \min\left\{r \in [0, +\infty),\ \text{s.t. } \partial\mathbb{B}_{\pi/2}(x_i) \subset \overline{\mathbb{B}}_r(p_i) \cup \overline{\mathbb{B}}_r(q_i)\right\}.$$

It is easy to check that $\frac{\pi}{2} \leqslant r_i \leqslant d_i < 2\pi$. We claim then

$$r_i \to \frac{\pi}{2}, \quad \text{as } i \to \infty. \tag{2}$$

If the claim is false, we can choose a subsequence, denoted always by $\{r_i\}$, which converges to $\frac{\pi}{2} + 2\delta$ with $\delta > 0$. So $\delta \leqslant \frac{\pi}{4}$ since $d_i \to \pi$. Therefore, for large enough $i,\ r_i > \frac{\pi}{2} + \delta$. Thanks to the definition of r_i, there exists a point $A_i \in \partial\mathbb{B}_{\pi/2}(x_i)$ such that

$$d(p_i, A_i) \geqslant \frac{\pi}{2} + \delta, \qquad d(q_i, A_i) \geqslant \frac{\pi}{2} + \delta.$$

Considering the following subsets in M_i,

$$\mathbb{B}_{\pi/2}(p_i),\ \mathbb{B}_{\pi/2}(q_i),\ \mathbb{B}_{\pi/2}(x_i) \text{ and } \mathbb{B}_\delta(A_i) \setminus \overline{\mathbb{B}}_{\pi/2}(x_i),$$

it is clear that they are pairwise disjoint. By V.C.T., we get that

$$\begin{aligned}\left(\frac{3}{2}+\eta_i\right)V\left(S^{2n}\right) &\geqslant V(M_i) \geqslant V\left(\mathbb{B}_{\pi/2}(p_i)\right) + V\left(\mathbb{B}_{\pi/2}(q_i)\right) + V\left(\mathbb{B}_{\pi/2}(x_i)\right) + V\left(\mathbb{B}_\delta(A_i) \setminus \mathbb{B}_{\pi/2}(x_i)\right)\\ &\geqslant \frac{3}{2}V\left(S^{2n}\right) + V\left(\mathbb{B}^s_{\delta/2}\right).\end{aligned}$$

This gives a contradiction since $\eta_i \to 0$ as $i \to \infty$. So (2) is true. Setting now

$$R_i = \max\left\{r_i, d_i - \frac{\pi}{2}\right\},$$

it is clear that $R_i \to \frac{\pi}{2}$ as $i \to \infty$. By the definitions of r_i and R_i,

$$\partial\mathbb{B}_{\pi/2}(x_i) \subset \overline{\mathbb{B}}_{R_i}(p_i) \cup \overline{\mathbb{B}}_{R_i}(q_i)$$

and

$$\partial\mathbb{B}_{\pi/2}(x_i) \cap \overline{\mathbb{B}}_{R_i}(p_i) \neq \emptyset, \qquad \partial\mathbb{B}_{\pi/2}(x_i) \cap \overline{\mathbb{B}}_{R_i}(q_i) \neq \emptyset.$$

Due to the connectivity of $\partial\mathbb{B}_{\pi/2}(x_i)$, we get

$$\partial\mathbb{B}_{\pi/2}(x_i) \cap \overline{\mathbb{B}}_{R_i}(p_i) \cap \overline{\mathbb{B}}_{R_i}(q_i) \neq \emptyset.$$

There exists then $A_i \in \partial\mathbb{B}_{\pi/2}(x_i)$ such that

$$\frac{\pi}{2} \leqslant d(A_i, p_i) \leqslant R_i \quad \text{and} \quad \frac{\pi}{2} \leqslant d(A_i, q_i) \leqslant R_i.$$

This means

$$d(A_i, x_i) = \frac{\pi}{2} \quad \text{and} \quad \lim_{i\to\infty} d(A_i, p_i) = \lim_{i\to\infty} d(A_i, q_i) = \frac{\pi}{2}. \tag{3}$$

Denoting by α_i the angle at A_i between γ_{A_i,p_i} and γ_{A_i,x_i}, we consider the geodesic triangle $(p_i A_i x_i)$ in M_i and a corresponding geodesic triangle $(\tilde{p}_i \tilde{A}_i \tilde{x}_i)$ in $\mathbb{R}^2$ which satisfies

$$d(A_i, p_i) = d_e\big(\tilde{A}_i, \tilde{p}_i\big),\ d(A_i, x_i) = d_e\big(\tilde{A}_i, \tilde{x}_i\big) \quad \text{and} \quad \alpha_i = \tilde{\alpha}_i,$$

where $d_e(x, y)$ denotes the distance between x and y in $\mathbb{R}^2$, $\tilde{\alpha}_i$ denotes the angle at $\tilde{A}_i$ between the segments $\tilde{A}_i\tilde{p}_i$ and $\tilde{A}_i\tilde{x}_i$ in $\mathbb{R}^2$. Applying Toponogov Comparison Theorem (see [5], p. 162), we have that $d(p_i, x_i) \leqslant d_e(\tilde{p}_i, \tilde{x}_i)$. By the Law of Cosine in $\mathbb{R}^2$, we obtain

$$\begin{aligned} d^2(p_i, x_i) \leqslant d_e^2(\tilde{p}_i, \tilde{x}_i) &= d_e^2\big(\tilde{A}_i, \tilde{p}_i\big) + d_e^2\big(\tilde{A}_i, \tilde{x}_i\big) - 2d_e\big(\tilde{A}_i, \tilde{p}_i\big)d_e\big(\tilde{A}_i, \tilde{x}_i\big)\cos\tilde{\alpha}_i \\ &= d^2(p_i, A_i) + d^2(x_i, A_i) - 2d(A_i, p_i)d(A_i, x_i)\cos\alpha_i. \end{aligned}$$

On the other hand, up to a subsequence, we can suppose that $\lim_{i\to\infty}\alpha_i = \alpha$. Using (3) and $\lim_{i\to\infty} d(p_i, x_i) = \pi$, we obtain, by passing i to ∞,

$$\pi^2 \leqslant \left(\frac{\pi}{2}\right)^2 + \left(\frac{\pi}{2}\right)^2 - \frac{\pi^2}{2}\cos\alpha.$$

This means $\cos\alpha = -1$, i.e., $\alpha = \pi$. Similarly, if we set β_i is the angle at point A_i between γ_{A_i,q_i} and γ_{A_i,x_i} and consider the geodesic triangle $(q_i A_i x_i)$, up to a subsequence, we can suppose that $\lim_{i\to\infty}\beta_i = \beta$, then we have $\beta = \pi$. Combining the argument above, we have

$$d(p_i, q_i) \to 0 \quad \text{as } i \to \infty.$$

This contradicts to the fact that $d(p_i, q_i) \to \pi$. The proof is completed.

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