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Deformation of big pseudoholomorphic disks and application to the Hanh pseudonorm

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Abstract

We simplify the proof of the theorem that close to any pseudoholomorphic disk there passes a pseudoholomorphic disk of arbitrary close size with any pre-described sufficiently close direction. We apply these results to the Kobayashi and Hanh pseudodistances. It is shown they coincide in dimensions higher than four. The result is new even in the complex case. ***To cite this article: B. Kruglikov, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Déformation de grands disques pseudo-holomorphes et application à la pseudonorme de Hanh. Nous simplifions la preuve du théorème montrant que près de tout disque pseudo-holomorphe il passe un disque pseudo-holomorphe de taille proche quelconque et avec une direction pré-fixée suffisamment proche. Nous appliquons ces résultats aux pseudodistances de Kobayashi et Hanh. Nous montrons qu'elles coïncident en dimensions supérieures à quatre. Le résultat est nouveau, même dans le cas complexe. ***Pour citer cet article : B. Kruglikov, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

Théorème 0.1. *Pour une variété quasi-complexe (M, J), considérons un disque pseudo-holomorphe*

$$f_0 : (D_R, i) \to (M, J), \quad (f_0)_*(0)e = v_0 \neq 0.$$

Ici, $e = 1$ est le vecteur unité en $0 \in \mathbb{C}$. Pour tout $\varepsilon > 0$, il existe un voisinage $\mathcal{V}_\varepsilon(v_0)$ du vecteur $v_0 \in TM$ tel que à chaque $v \in \mathcal{V}_\varepsilon(v_0)$ corresponde un disque pseudo-holomorphe

$$f : (D_{R-\varepsilon}, i) \to (M, J), \quad f_*(0)e = v.$$

La courbe approximante f peut être choisie plongée/immergée si la courbe f_0 le peut.

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Ce théorème a été prouvé dans [2] par la méthode de Newton et la machinerie de [7] . Nous le prouvons ici grâce au théorème des fonctions implicites pour l'équation linéarisée le long du disque pseudo-holomorphe, en exploitant l'opérateur de Green T_r. L'assertion généralise les Théorèmes 1.7, 3.1.1(ii) de [5] et [10] respectivement.

Une application est reliée à la pseudo-norme de Kobayashi–Royden : $F_M(v) = \inf\{1/r \mid f:(D_r, i) \to (M, J),\ f_*(0)e = v\}$, $v \in TM$. Le théorème ci-dessus assure la semi-continuité supérieure de F_M, ce qui implique que la pseudo-distance de Kobayashi $d_M(x, y) = \inf\{\int \gamma^* F_M \mid \gamma : [0, 1] \to M,\ \gamma(0) = x,\ \gamma(1) = y\}$ est bien définie. De façon similaire, la pseudo-norme de Hanh $S_M(v)$ est définie avec la condition supplémentaire que f est injective. Cela engendre aussi la pseudo-distance de Hanh h_M via l'intégration suivant un chemin. Chaque pseudo-distance coïncide avec celle définie grâce aux chaînes de disques pseudo-holomorphes (resp. disques injectifs) [2].

Théorème 0.2. *Pour les variétés quasi-complexes* (M^{2n}, J)*,* $n > 2$*, on a :* $S_M = F_M$.

Dans le cas de domaines $M \subset \mathbb{C}^n$, la formule a été prouvée dans [8]. La généralisation est nouvelle, même dans le cas complexe général. La preuve est basée sur une idée de type Whitney pour déformer la courbe, avec un nombre suffisant de paramètres pour résoudre les singularités.

1. Deformation of big pseudoholomorphic disks

We aim here to prove the following statement, which was proved by another (analogous to the approach of [7]) and more complicated method in [2].

Theorem 1.1. *Let* (M^{2n}, J) *be an almost complex manifold and*

$$f_0 : (D_R, i) \to (M, J), \quad (f_0)_*(0)e = v_0 \in TM, \quad v_0 \neq 0,$$

be a pseudoholomorphic disk. Here $e = 1$ *is the unit vector at* $0 \in \mathbb{C}$*. For every* $\varepsilon > 0$ *there exists a neighborhood* $\mathcal{V}_\varepsilon(v_0)$ *of the vector* v_0 *such that for each* $v \in \mathcal{V}_\varepsilon$ *there is an* ε*-close in a fixed* C^k*-norm, slightly smaller pseudoholomorphic disk*

$$f : (D_{R-\varepsilon}, i) \to (M, J), \quad f_*(0)e = v.$$

The approximating curve f *can be embedded/immersed if such is the curve* f_0.

This theorem was used in [2] for the proof of equivalence of two definitions of Kobayashi pseudodistance d_M in almost complex category. In the second definition d_M is associated via path integration to the Kobayashi–Royden pseudonorm:

$$F_M(v) = \inf\bigl\{1/r \mid f : (D_r, i) \to (M, J),\ f_*(0)e = v\bigr\}, \quad v \in TM.$$

The above theorem assures F_M to be upper semicontinous, implying that

$$d_M(x, y) = \inf\left\{ \int \gamma^* F_M \mid \gamma : [0, 1] \to M,\ \gamma(0) = x,\ \gamma(1) = y \right\}$$

is well-defined.

Moreover, since an embedded disk can always be perturbed to embedded we prove simultaneously the main properties of the Hanh pseudonorm $S_M(v)$, which is defined by the same formula as F_M with an additional requirement on f to be injective. This pseudometric generates a pseudodistance via path integration, like F_M generates d_M, and this coincides (cf. [2]) with the distance $h_M(x, y) = \inf \sum_{k=1}^m d(z_k, w_k)$, defined via injective chains $f_k : D_1 \to (M^{2n}, J)$, $k = 1, \dots, m$, $f_1(z_1) = p$, $f_m(w_m) = q$ and $f_k(w_k) = f_{k+1}(z_{k+1})$, where d is the Poincaré metric on D_1.

2. New proof of Theorem 1.1

Our approach to the close pseudoholomorphic curves existence result is similar to that of [10], where the linearization of the structure J was made at a point. We linearize the structure along the disk and use the reduction of the almost complex problem to a complex one via the Green operator:

$$T_r : C^k\big(D_r, \mathbb{C}^n\big) \to C^{k+1}\big(D_r, \mathbb{C}^n\big), \qquad g(z) \mapsto \frac{1}{2\pi \mathrm{i}} \iint\limits_{D_r} \frac{g(z)}{\zeta - z}\, \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}.$$

It is continuous in the Sobolev and Hölder norms [11] and obeys the identities: $\bar\partial T_r = \mathrm{Id}$, $T_r \bar\partial|_{C_0^{k+1}} = \mathrm{Id}$.

Proof. We study at first the case, when the curve is embedded. Let U be a neighborhood of the shrunk pseudoholomorphic curve $f_0(D_{R-\varepsilon})$. We can assume [2] the disk is standard $f_0(D_{R-\varepsilon}) = D_{R-\varepsilon} \times \{0\}^{n-1} \subset \mathbb{C}^n$ and the almost complex structure $J : U \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)$, $J^2 = -\mathbf{1}$, along it is the standard complex structure $J(z) = J_0$ for all $z \in D_{R-\varepsilon}$. The equation for f to be pseudoholomorphic reads:

$$\bar\partial f + q_J(f)\partial f = 0, \qquad q_J(z) = \big[J_0 + J(z)\big]^{-1} \cdot \big[J_0 - J(z)\big],$$

which due to the above properties is equivalent to

$$\bar\partial h = 0, \quad h = \big[\mathrm{Id} + T_{R-\varepsilon} \circ q_J(f) \circ \partial\big](f).$$

For $k \in \mathbb{R} \setminus \mathbb{Z}$, $k > 1$, consider the map

$$\begin{aligned} \Phi : \mathcal{J} \times C^{k+1}(D_{R-\varepsilon}; U) &\longrightarrow C^{k+1}\big(D_{R-\varepsilon}; \mathbb{C}^n\big), \\ (J, s) &\longmapsto \big[\mathrm{Id} + T_{R-\varepsilon} \circ q_J(f_0 + s) \circ \partial\big](f_0 + s), \end{aligned}$$

where $\mathcal{J}$ is a neighborhood of the given almost complex structure J in C^k-topology. We consider U as the total space of the "normal bundle", with the sections being denoted by s, so that every map $f \in C^{k+1}(D_{R-\varepsilon}; U)$, that is C^1-close to $(f_0)|_{D_{R-\varepsilon}}$, has a unique representation $f = f_0 + s$.

The map $\Phi_J = \Phi(J, \cdot)$ is C^k-smooth and satisfies: $\Phi_J(0) = f_0$, $\Phi_J'(0) = \mathrm{Id}$. It has the Taylor decomposition (with $\|\cdot\|$ being the C^{k+1}-norm):

$$\Phi_J(s) = f_0 + s + \mathrm{o}\big(\|s\|\big).$$

Therefore $\mathrm{Im}\, \Phi_J$ contains a small neighborhood of the curve f_0.

Let $Z = (a, v) \in T\mathbb{C}^n$ and $h_Z(z) = a + vz$ be the holomorphic disk in U, $z \in D_{R-\varepsilon}$. It is close to f_0 whenever Z is close to $Z_0 = (0, (1, 0, \ldots, 0)) \in T\mathbb{C}^n$. Define

$$f_Z = f_0 + \Phi_J^{-1}(h_Z).$$

It is a J-holomorphic $(R - \varepsilon)$-disk, which satisfies: $f_Z - h_Z = \mathrm{o}(|Z - Z_0|)$.

Consider the C^k-map $\Psi : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$, $Z \mapsto (f_Z(0), (f_Z)_*(0)e)$. Since the above estimate implies $\Psi'(Z_0) = \mathrm{Id}$, the map $\Psi(Z)$ is a local C^k-diffeomorphism of a neighborhood of Z_0. In particular, for every $Z = (a, v)$ sufficiently close to Z_0 there exists a pair $\widetilde{Z} = (\alpha, \zeta)$ such that $\Psi(\widetilde{Z}) = Z$.

Now the obtained map $f = f_{\widetilde{Z}}$ is C^1-close to f_0 and so is embedded. It is also smooth due to the usual elliptic regularity [7,5,10]. If f_0 is immersed, the reasoning is the same for the neighborhood U obtained via f_0 by the pull-back.

In the general case for the map $f_0 : (D_{R-\varepsilon}, i) \to (M, J)$ we consider the graph $\hat{f}_0 : (D_{R-\varepsilon}, i) \to (D_{R-\varepsilon} \times M, \hat{J} = i \times J)$, which is injective and apply the part of the statement already proved. □

Remark 1. The proof implies persistence of big pseudoholomorphic disks (with an insignificant loss of size) under perturbation not only of the initial vector, but also of the almost complex structure J (note the role of $\mathcal{J}$ above), as well as existence of a deformation of the initial curve to the perturbed one. This generalizes Theorems 1.7 of [5] and 3.1.1(ii) of [10].

3. Kobayashi–Royden vs. Hanh pseudometric

The properties of the Kobayashi–Royden pseudometric for almost complex manifolds was discussed in [2]. Let us consider the non-integrable version S_M of the Hanh pseudometric. By a theorem of Overholt [8] it coincides with the Kobayashi–Royden pseudometric F_M for domains $M \subset \mathbb{C}^n$ of dimension $n > 2$. We generalize this to the non-integrable case.

Theorem 3.1. *$S_M = F_M$ for almost complex manifolds (M^{2n}, J), $n > 2$.*

Proof. Since $S_M \geqslant F_M$, it is enough to show that whatever small $\varepsilon > 0$ is, any pseudoholomorphic disk of radius $R > 0$ can be approximated by an injective pseudoholomorphic disk of radius $R - \varepsilon$ with the same initial direction.

We give at first a new simple proof of Overholt's theorem from [8]. Let $M \subset \mathbb{C}^n$ be a domain and $f : D_R \to M$ be a holomorphic map. Denote $f_W(z) = f(z) - w_2 z^2 - w_3 z^3$, $z \in D_{R-\varepsilon}$, $W = (w_2, w_3) \in \mathbb{C}^{2n}$. For small W the map has still the image in M. Also note that $f_W(0) = f(0)$ and $f'_W(0) = f'(0)$.

By the Sard theorem a generic $w_2 \in \mathbb{C}^n$ is outside the set

$$\left\{ \frac{f(z_1) - f(z_2)}{z_1^2 - z_2^2} \,\middle|\, z_1, z_2 \in D_{R-\varepsilon} \right\} \cup \left\{ \frac{f'(z)}{2z} \,\middle|\, z \in D_{R-\varepsilon} \right\}.$$

For such a choice the map $f_{w_2,0}$ is injective outside the anti-diagonal $\{z_2 = -z_1\}$. Note that regularity of the origin is preserved. So, switching on w_3 being generic, we get the map f_{w_2,w_3} to be injective everywhere.

In other words, the Sard theorem implies that the set of $W = (w_2, w_3)$ for which f_W is not injective has measure zero and so a generic pair of small vectors $w_2, w_3 \in \mathbb{C}^n$ defines the required approximating disk $f_W(z)$.

In the general complex case we should shift along some holomorphic vector fields. This is achieved by the graph-lift construction and Royden's lemma [9] that an embedded holomorphic disk, shrunk a bit, has a Stein neighborhood.

It is easier, however, to consider the general case of almost complex manifolds (M, J) and to deduce the statement for integrable J as a corollary.

Denote by $\pi : D_{R-\varepsilon} \times M \to M$ the projection. As in Theorem 1.1, the graph-lift $\hat{f}_0 : D_{R-\varepsilon} \to D_{R-\varepsilon} \times M$ can be deformed to the family $\hat{f}_{\widehat{W}} = \hat{f}_0 + \hat{\Phi}_{\hat{J}}^{-1}(g_{\widehat{W}})$, where $g_{\widehat{W}} = \widehat{w}_0 + \widehat{w}_1 z - \widehat{w}_2 z^2 - \widehat{w}_3 z^3$, $\widehat{W} = (\widehat{w}_2, \widehat{w}_3)$, $\widehat{w}_j \in \mathbb{C}^{n+1}$ and $(\widehat{w}_0, \widehat{w}_1) = (\hat{\varphi}_0(\widehat{w}_2, \widehat{w}_3), \hat{\varphi}_1(\widehat{w}_2, \widehat{w}_3))$ are some C^k-smooth functions, close to $\widehat{Z}_0 = (0, (1, 0, \ldots, 0))$ and such that $(\hat{f}_{\widehat{W}}(0), (\hat{f}_{\widehat{W}})_*(0)e) = \widehat{Z}_0 \in T\mathbb{C}^{n+1}$. We identify above $\hat{f}_0$ with $g_{\hat{0}}$, the first coordinate disk, and its neighborhood with a ball $B \subset \mathbb{C}^{n+1}$, equipped with the structure $\hat{J} = i \times J$.

Similarly to the first proof we get: $\hat{f}_{\widehat{W}} = g_{\widehat{W}} + \rho_{\widehat{W}}$, where $\rho_{\widehat{W}} = \mathrm{o}(|\widehat{W}|)$. Now $f_{\widehat{W}}$ is an embedding if $\pi \hat{f}_{\widehat{W}}(z_1) \neq \pi \hat{f}_{\widehat{W}}(z_2)$ for $z_1 \neq z_2$ and $\partial \pi \hat{f}_{\widehat{W}}(z) \neq 0$. We consider only the first, more complicated, injectivity condition. It's negation is equivalent to $g_{\widehat{W}}(z_1) - g_{\widehat{W}}(z_2) = [\rho_{\widehat{W}}]|_{z_1}^{z_2} + \zeta$, $\zeta \in D$, or equivalently:

$$\widehat{w}_2(z_1 + z_2) + \widehat{w}_3\big(z_1^2 + z_1 z_2 + z_2^2\big) = \widehat{w}_1 + \frac{\rho_{\widehat{W}}(z_2) - \rho_{\widehat{W}}(z_1)}{z_2 - z_1} + \tilde{\zeta}.$$

The last equation is never satisfied for a.e. small $\widehat{W} = (\widehat{w}_2, \widehat{w}_3)$ in $\mathbb{C}^{2n+2}$. In fact, for $\widehat{w}_1 = \varphi_1(\widehat{w}_2, \widehat{w}_3)$ the r.h.s. is $\mathrm{o}(|\widehat{W}|)$. Thus the claim follows from the Sard theorem, if at least one of the coefficients of $\widehat{w}_2$ and $\widehat{w}_3$ is not small. Since

$$D \times D = \big[U_{5\delta}(z_1 = z_2 = 0)\big] \cup \big[D \times D \setminus U_\delta(z_1 = -z_2)\big] \cup \big[D \times D \setminus U_\delta\big(z_1 = \big(-\tfrac{1}{2} \pm \mathrm{i}\tfrac{\sqrt{3}}{2}\big) z_2\big)\big], \tag{1}$$

and the regularity at $(0, 0)$ is preserved under small perturbation we may achieve injectivity away from the anti-diagonal by the quadratic perturbation and then in its neighborhood by a cubic one. This finishes the proof. □

For $n = 1$, when almost complex structures are automatically integrable, the equality $S_M = F_M$ for domains $M \subset \mathbb{C}$ is equivalent to contractibility ($M = D_1$ or $\mathbb{C}$). In the case of $\mathbb{C}$-dimension $n = 2$ the equality may fail to hold (however arguments of Theorem 2 show that F_M coincides with the pseudonorm $\widetilde{S}_M$ obtained via immersed disks).

Example 1. Consider the map $\varphi_\alpha : D_1 \to \mathbb{C}^2$, $z \mapsto (z(\alpha z - 1)^2, \alpha z^2(\alpha z - 1))$, $|\alpha| > 1$. It has a unique self-intersection point $\varphi_\alpha(0) = \varphi_\alpha(1/\alpha) = 0$, which is transversal: $\varphi'_\alpha(0) = (1, 0)$, $\varphi_\alpha(1/\alpha) = (0, 1)$, and so non-removable. For a neighborhood U of the image $\mathrm{Im}(\varphi_\alpha)$ the pseudonorms F_U and S_U are different.

For the product $M^4 = U_1^2 \times U_2^2$ the pseudonorms S_M and F_M were compared in [1]. It is however unclear if we can majorize $S_M \leqslant c \cdot F_M$, with a constant depending on M, or more generally, if Kobayashi and Hanh hyperbolicities (d_M, resp. S_M being a metric) are equivalent. Of course, the former implies the latter.

It was shown in [4] that contractible tame almost complex domains are hyperbolic. In other cases the hyperbolicity may be lost.

Example 2. Consider the Reeb foliation of $\mathbb{R}^3$ with the standard T^2 as a leaf. This foliation propagates via parallel transports to $\mathbb{R}^{2n} = \mathbb{R}^3 \times \mathbb{R}^{2n-3}$, $n \geqslant 2$, and there is an almost complex J on $\mathbb{R}^{2n}$ making the foliation pseudoholomorphic. Every domain containing the leaf T^2 is neither tame nor hyperbolic. For $n = 2$ only a curve of genus 1 can be realized pseudoholomorphically in an almost complex $(\mathbb{R}^{2n}, J)$ [6]. For $n > 2$ the sphere S^2 can be realized pseudoholomorphically, yielding a non-tame and non-hyperbolic domain in $(\mathbb{R}^{2n}, J)$ [4].

Remark 2. In [6] the analogy between geodesics and pseudoholomorphic disks was exploited (the Nijenhuis tensor plays the role of the curvature [3]). It is however limited: by Theorem 3.1 there are no analogs for conjugate points in complex time curves theory. In fact the pseudoholomorphic curves are more flexible: there passes a pseudoholomorphic disk through every finite collection of points.

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