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Existence of a solution to a dynamic unilateral contact problem for a cracked viscoelastic body

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Abstract

In this paper we study a dynamic unilateral contact problem with friction for a cracked viscoelastic body. The viscoelastic model is characterized by Kelvin–Voigt's law and a nonlocal friction law is investigated here. The existence of a solution to the problem is obtained by using a penalty method. Several estimates are obtained on the solution to the penalized problem, which enable us to pass to the limit by using compactness results. **To cite this article:** *M. Cocou, G. Scarella, C. R. Acad. Sci. Paris, Ser. I 338 (2004).*

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Résumé

Existence d'une solution à un problème dynamique de contact unilatéral pour un milieu viscoélastique fissuré. Dans ce travail, on s'intéresse à un problème dynamique de contact unilatéral avec frottement non local pour un milieu viscoélastique fissuré, suivant une loi de comportement de Kelvin–Voigt. L'existence d'une solution du problème est obtenue par une méthode de pénalisation. Le passage à la limite est justifié par plusieurs estimations et par quelques résultats de compacité. **Pour citer cet article :** *M. Cocou, G. Scarella, C. R. Acad. Sci. Paris, Ser. I 338 (2004).*

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Peu de résultats d'existence ont été obtenus pour les problèmes dynamiques de contact unilatéral. Lebeau et Schatzman [7] ont étudié l'existence et l'unicité d'une solution de l'équation des ondes dans un demi-espace avec une condition unilatérale. Kim [5] a montré un résultat d'existence pour le même problème dans un ouvert borné. Petrov et Schatzman [10] ont étudié un problème unidimensionnel de contact unilatéral sans frottement dans un demi-espace, pour le modèle viscoélastique de Kelvin–Voigt. Dans le cas multidimensionnel, Jarušek [4] a obtenu un résultat d'existence pour un problème dynamique de contact unilatéral avec frottement, pour un milieu

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viscoélastique. Muñoz-Rivera et Racke [8] ont utilisé la même méthode que Kim [5] en thermoélasticité avec condition de symétrie radiale. Plus récemment, pour un matériau de Kelvin–Voigt, d’autres résultats d’existence ont été obtenus pour des problèmes dynamiques de contact avec frottement par Cocou, Ricaud et Pratt [2,3,11]. Paumier et Renard [9] ont résolu un problème de contact en élastodynamique avec perturbation de surface. Tous ces résultats ont été obtenus dans des domaines réguliers.

Dans ce travail, la méthode utilisée dans [3] est étendue à un milieu viscoélastique fissuré de Kelvin–Voigt. Dans une première partie, les équations du problème (1) et la formulation variationnelle associée (3) sont présentées, en utilisant une paramétrisation adaptée des lèvres de la fissure. Dans une deuxième partie, on considère une décomposition du domaine qui permet de considérer des espaces fonctionnels classiques. Une formulation variationnelle associée à des conditions aux limites obtenues par une pénalisation de l’interpénétration des deux lèvres de la fissure est donnée dans (4). Dans (5), une formulation variationnelle employant la décomposition du domaine, équivalente à (4), est introduite. Plusieurs estimations données dans (6) sont obtenues sur la solution du problème pénalisé (5). A partir de celles-ci, ainsi que de résultats de compacité, on montre qu’une sous-suite de solutions pénalisées converge faiblement vers une solution du problème de contact unilatéral.

1. Introduction

Few results deal with the existence of a solution to unilateral contact problems in dynamics. Lebeau and Schatzman [7] proved existence and uniqueness results for a solution to a wave equation in a half space with frictionless unilateral boundary conditions. Kim [5] showed the existence of a solution to the same problem for a smooth bounded domain. Petrov and Schatzman [10] have studied a one-dimensional unilateral contact problem in viscoelasticity with Kelvin–Voigt’s model, in a half space. For multidimensional cases, Jarušek [4] was the first to obtain an existence result to dynamic unilateral contact problems with friction, for viscoelastic bodies. Muñoz-Rivera and Racke [8] employed the same method as Kim [5] in thermoelasticity with radial symmetry. More recently, in the case of a Kelvin–Voigt’s material, Cocou and Ricaud [2] have proved existence results for dynamic contact problems with nonlocal friction, under boundedness hypotheses on velocity and acceleration. Pratt and Ricaud [11] have studied dynamic contact problems with friction by using discretization techniques, in a particular case of a normal compliance law. Paumier and Renard [9] have solved an elastodynamic contact problem with surface perturbation. In this paper, we extend the existence result obtained in [3], for a dynamic unilateral contact problem with nonlocal friction, without any additional assumption on velocity or acceleration, to a cracked viscoelastic body. Firstly, we give classical and variational formulations of the problem. Then a variational formulation of a penalized problem is considered. An equivalent formulation using a decomposition of the domain is presented for the penalized problem. Several estimates allow us to pass to the limit by using compactness arguments.

2. Formulations of the problem

Let Ω be a bounded domain of \mathbb{R}^d , $d = 2$ or 3 . A cracked viscoelastic body satisfying Kelvin–Voigt’s law initially occupies Ω . In this study, we assume the small deformation hypothesis. The body is subjected to volume forces of density \mathbf{f} and its boundary, which is denoted by $\partial\Omega$, is composed of three parts such that $\partial\Omega = \bar{\Gamma}_U \cup \bar{\Gamma}_F \cup \bar{\Gamma}$, where Γ_U , Γ_F are sufficiently smooth with $\text{meas}(\Gamma_U) > 0$. Zero displacements are imposed on Γ_U , prescribed tractions \mathbf{F} are imposed on Γ_F and the crack is denoted by Γ . We suppose that Γ is composed of two parts, $\bar{\Gamma} = \bar{\Gamma}^+ \cup \bar{\Gamma}^-$, which may be in unilateral contact with friction. We consider a nonlocal Coulomb’s friction law corresponding to a regularization \mathcal{R} which is defined later. Using a similar technique to the one proposed by Boieri et al. [1] to formulate contact conditions between two bodies, we choose an open subset \mathcal{E} of \mathbb{R}^{d-1} to parametrize the faces of the crack and to express boundary conditions. The notation $[\cdot]$ denotes the

jump between the upper part of the crack Γ^+ and the lower part Γ^- . We define $[w_N] = \mathbf{w}^+ \cdot \mathbf{n}^+ + \mathbf{w}^- \cdot \mathbf{n}^-$, $[\mathbf{w}_T] = \mathbf{w}_T^+ - \mathbf{w}_T^-$, $\sigma_N^\alpha = \boldsymbol{\sigma} \mathbf{n}^\alpha \cdot \mathbf{n}^\alpha$, $\boldsymbol{\sigma}_T^\alpha = \boldsymbol{\sigma} \mathbf{n}^\alpha - \sigma_N^\alpha \mathbf{n}^\alpha$ and σ_N^α is the normal component of $\boldsymbol{\sigma} \mathbf{n}^\alpha$, where \mathbf{n}^α is the unit outward normal to Γ^α , $\alpha = +, -$. We denote respectively by g and μ the gap between the faces of the crack and the friction coefficient. Let $\mathcal{A} = (\mathcal{A}_{ijkl})$, $\mathcal{B} = (\mathcal{B}_{ijkl})$ denote two fourth-order tensors which satisfy classical symmetry and ellipticity conditions such that $\mathcal{A}_{ijkl}, \mathcal{B}_{ijkl} \in W^{1,\infty}(\mathbb{R}^d)$, $\forall i, j, k, l = 1, \dots, d$. The problem is the following.

Problem P₀. Find $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ such that $\mathbf{u}(0) = \mathbf{u}_0$, $\dot{\mathbf{u}}(0) = \mathbf{u}_1$ in Ω and

$$\left\{ \begin{array}{l} \ddot{\mathbf{u}} - \operatorname{div} \boldsymbol{\sigma}(\mathbf{u}, \dot{\mathbf{u}}) = \mathbf{f}, \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{u}, \dot{\mathbf{u}}) = \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}) + \mathcal{B}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}) \quad \text{in }]0, T[\times \Omega, \\ \mathbf{u} = \mathbf{0} \quad \text{in }]0, T[\times \Gamma_U, \quad \boldsymbol{\sigma} \mathbf{n} = \mathbf{F} \quad \text{in }]0, T[\times \Gamma_F, \\ [u_N] \leq g, \quad \sigma_N^- = \sigma_N^+ \leq 0, \quad \sigma_N^-([u_N] - g) = 0 \quad \text{in }]0, T[\times \mathcal{E}, \\ \boldsymbol{\sigma}_T^+ = -\boldsymbol{\sigma}_T^- \quad \text{in }]0, T[\times \mathcal{E}, \\ \|\boldsymbol{\sigma}_T^+\| \leq \mu |(\mathcal{R}\boldsymbol{\sigma})_N^+| \quad \text{and} \quad \|\boldsymbol{\sigma}_T^+\| < \mu |(\mathcal{R}\boldsymbol{\sigma})_N^+| \Rightarrow [\dot{\mathbf{u}}_T] = \mathbf{0}, \\ \|\boldsymbol{\sigma}_T^+\| = \mu |(\mathcal{R}\boldsymbol{\sigma})_N^+| \Rightarrow \exists \lambda \geq 0, [\dot{\mathbf{u}}_T] = -\lambda \boldsymbol{\sigma}_T^+. \end{array} \right. \tag{1}$$

Let us first introduce the following functional spaces \mathbf{H}, \mathbf{V} and the set \mathbf{K} : $\mathbf{H} = [L^2(\Omega)]^d$, $\mathbf{V} = \{\mathbf{v} \in [H^1(\Omega)]^d; \mathbf{v} = \mathbf{0} \text{ a.e. on } \Gamma_U\}$, $\mathbf{K} = \{\mathbf{v} \in \mathbf{V}; [v_N] \leq g \text{ a.e. in } \mathcal{E}\}$.

We assume that $\mathbf{u}_0 \in \mathbf{K}$, $\mathbf{u}_1 \in \mathbf{V}$, $\mathbf{f} \in W^{1,\infty}(0, T; \mathbf{H})$, $\mathbf{F} \in W^{1,\infty}(0, T; [L^2(\Gamma_F)]^d)$, $g \in H_{00}^{1/2}(\mathcal{E})$, $g \geq 0$, $\mu \in L^\infty(\mathcal{E})$. The particular case $g = 0$ corresponds to a cut. A nonlocal friction law of the same type as in [3,6] is investigated here. We suppose that $\mathcal{R}: [L^2_{\text{sym}}(\Omega)]^{d^2} \rightarrow [H^1(\Omega)]^{d^2}$ is a linear and continuous regularization of $\boldsymbol{\sigma}$. We also assume that $((\mathcal{R}\boldsymbol{\sigma})(\mathbf{u}_0, \mathbf{u}_1))_N^+ = 0$ and

$$\exists C > 0, \quad \|(\mathcal{R}\boldsymbol{\sigma})(\mathbf{u}, \mathbf{v})\|_{\mathbf{V}} \leq C(\|\mathbf{u}\|_{\mathbf{H}} + \|\mathbf{v}\|_{\mathbf{H}}), \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{V}. \tag{2}$$

Let us define two bilinear, continuous and symmetric mappings a, b on $\mathbf{V} \times \mathbf{V} \rightarrow \mathbb{R}$, which correspond to the elastic tensor and to the viscous tensor, and the mapping J as follows:

$$\begin{aligned} a(\mathbf{v}, \mathbf{w}) &= \int_{\Omega} \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\varepsilon}(\mathbf{w}) \, dx, & b(\mathbf{v}, \mathbf{w}) &= \int_{\Omega} \mathcal{B}\boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\varepsilon}(\mathbf{w}) \, dx, \\ J : \mathbf{V} \times \mathbf{V} \times \mathbf{V} &\rightarrow \mathbb{R}, & J(\mathbf{u}, \mathbf{v}, \mathbf{w}) &= \int_{\mathcal{E}} \mu |((\mathcal{R}\boldsymbol{\sigma})(\mathbf{u}, \mathbf{v}))_N^+| \|[\mathbf{w}_T]\| \, d\xi, \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}. \end{aligned}$$

Let us denote by \mathbf{L} the element of \mathbf{V}' such that $\langle \mathbf{L}, \mathbf{v} \rangle = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx + \int_{\Gamma_F} \mathbf{F} \cdot \mathbf{v} \, ds$, $\forall \mathbf{v} \in \mathbf{V}$. We assume the following compatibility condition: $\exists \mathbf{l} \in \mathbf{H}$, $a(\mathbf{u}_0, \mathbf{v}) + b(\mathbf{u}_1, \mathbf{v}) = \langle \mathbf{L}(0), \mathbf{v} \rangle - \langle \mathbf{l}, \mathbf{v} \rangle$, $\forall \mathbf{v} \in \mathbf{V}$. Now, we give a variational formulation of problem P_0 .

Problem P₁. Find $\mathbf{u} \in W^{1,2}(0, T; \mathbf{V}) \cap C^1([0, T]; [H^{-1/2}(\Omega)]^d)$ such that $\mathbf{u}(t) \in \mathbf{K}$ for all $t \in]0, T[$ and

$$\left\{ \begin{array}{l} \langle \dot{\mathbf{u}}(T), \mathbf{v}(T) - \mathbf{u}(T) \rangle_{H^{-1/2}, H^{1/2}} - \langle \mathbf{u}_1, \mathbf{v}(0) - \mathbf{u}_0 \rangle \\ + \int_0^T \{ -\langle \dot{\mathbf{u}}, \dot{\mathbf{v}} \rangle + a(\mathbf{u}, \mathbf{v}) + b(\dot{\mathbf{u}}, \mathbf{v}) + J(\mathbf{u}, \dot{\mathbf{u}}, \mathbf{v} + \dot{\mathbf{u}} - \mathbf{u}) \} \, dt \\ \geq \int_0^T \langle \mathbf{L}, \mathbf{v} - \mathbf{u} \rangle \, dt + \int_0^T \{ -\|\dot{\mathbf{u}}\|_{\mathbf{H}}^2 + a(\mathbf{u}, \mathbf{u}) + b(\dot{\mathbf{u}}, \mathbf{u}) + J(\mathbf{u}, \dot{\mathbf{u}}, \dot{\mathbf{u}}) \} \, dt \\ \forall \mathbf{v} \in L^\infty(0, T; \mathbf{V}) \cap W^{1,2}(0, T; \mathbf{H}), \quad \mathbf{v}(t) \in \mathbf{K} \text{ a.e. } t \in]0, T[. \end{array} \right. \tag{3}$$

3. The penalized problems

We consider a penalized viscoelastic contact problem with a solution denoted by u_ε verifying the same equations in Ω and the same boundary conditions on $\Gamma_U, \Gamma_F, \Gamma$ as before, except for unilateral contact conditions. The penetration between the two faces of the crack is penalized. For almost every $t \in]0, T[$, contact conditions on \mathcal{E} are the following, using the same friction conditions as before: $\sigma_N^+(u_\varepsilon, \dot{u}_\varepsilon) = \sigma_N^-(u_\varepsilon, \dot{u}_\varepsilon) = -(1/\varepsilon) ([u_{\varepsilon N}] - g)_+$, where $r_+ = \max(0, r)$. We define $\Phi_\varepsilon : V \times V \rightarrow \mathbb{R}$ by $\Phi_\varepsilon(v, w) = (1/\varepsilon) \int_{\mathcal{E}} ([v_N] - g)_+[w_N] ds$. A variational formulation of this problem is the following.

Problem P_2^ε . Find $u_\varepsilon \in W^{1,2}(0, T; V) \cap W^{2,2}(0, T; H)$ such that $u_\varepsilon(0) = u_0, \dot{u}_\varepsilon(0) = u_1$, and

$$\begin{cases} (\ddot{u}_\varepsilon, w - \dot{u}_\varepsilon) + a(u_\varepsilon, w - \dot{u}_\varepsilon) + b(\dot{u}_\varepsilon, w - \dot{u}_\varepsilon) + \Phi_\varepsilon(u_\varepsilon, w - \dot{u}_\varepsilon) \\ + J(u_\varepsilon, \dot{u}_\varepsilon, w) - J(u_\varepsilon, \dot{u}_\varepsilon, \dot{u}_\varepsilon) \geq \langle L, w - \dot{u}_\varepsilon \rangle, \quad \forall w \in V, \text{ a.e. } t \in]0, T[. \end{cases} \tag{4}$$

We suppose that we can decompose Ω into two open disjoint subsets Ω^+ and Ω^- with Lipschitz continuous boundaries $\partial\Omega^+$ and $\partial\Omega^-$, respectively, such that $\Omega = \Omega^+ \cup \Omega^- \cup \Gamma_V$, with $\Gamma_V \subset \partial\Omega^+ \cap \partial\Omega^-$, $\Gamma^\alpha = \Gamma \cap \partial\Omega^\alpha$ and $\text{meas}(\Gamma_V^\alpha) > 0$, where $\Gamma_V^\alpha = \Gamma_U \cap \partial\Omega^\alpha$, $\alpha = +, -$. This decomposition enables us to reduce the variational analysis to more classical domains. For $w \in [L^2(\Omega)]^d$, we set $\hat{w} = (w^+, w^-)$, where w^α is the restriction of w on Ω^α and we introduce the following functional spaces:

$$\begin{aligned} H^\alpha &= [L^2(\Omega^\alpha)]^d, \quad \hat{H} = H^+ \times H^-, \quad V^\alpha = \{v \in [H^1(\Omega^\alpha)]^d; v = 0 \text{ a.e. on } \Gamma_U^\alpha\}, \quad \alpha = +, -, \\ \hat{V} &= \{\hat{v} = (v^+, v^-) \in V^+ \times V^-; v^+ = v^- \text{ a.e. on } \Gamma_V\}. \end{aligned}$$

Note that \hat{V} is a closed subspace of $V^+ \times V^-$. An equivalent formulation of (4), based on the previous decomposition of Ω , is given as follows, where mappings $\hat{a}, \hat{b}, \hat{\Phi}_\varepsilon, \hat{J}, \hat{L}$, defined on \hat{V} , are equal to $a, b, \Phi_\varepsilon, J, L$ respectively, and the convex \hat{K} of \hat{V} corresponds to K .

Problem \hat{P}_2^ε . Find $\hat{u}_\varepsilon = (u_\varepsilon^+, u_\varepsilon^-) \in W^{1,2}(0, T; \hat{V}) \cap W^{2,2}(0, T; \hat{H})$ such that $\hat{u}_\varepsilon(0) = \hat{u}_0, \dot{\hat{u}}_\varepsilon(0) = \hat{u}_1$,

$$\begin{cases} (\ddot{\hat{u}}_\varepsilon, \hat{w} - \dot{\hat{u}}_\varepsilon) + \hat{a}(\hat{u}_\varepsilon, \hat{w} - \dot{\hat{u}}_\varepsilon) + \hat{b}(\dot{\hat{u}}_\varepsilon, \hat{w} - \dot{\hat{u}}_\varepsilon) + \hat{\Phi}_\varepsilon(\hat{u}_\varepsilon, \hat{w} - \dot{\hat{u}}_\varepsilon) \\ + \hat{J}(\hat{u}_\varepsilon, \dot{\hat{u}}_\varepsilon, \hat{w}) - \hat{J}(\hat{u}_\varepsilon, \dot{\hat{u}}_\varepsilon, \dot{\hat{u}}_\varepsilon) \geq \langle \hat{L}, \hat{w} - \dot{\hat{u}}_\varepsilon \rangle, \quad \forall \hat{w} \in \hat{V}, \text{ a.e. } t \in]0, T[. \end{cases} \tag{5}$$

The method used, e.g., in [11] enables us to prove that there exists a unique solution to this problem. By choosing $\hat{w} = 0$ in (5), we obtain

$$\begin{aligned} \frac{1}{2} \|\dot{\hat{u}}_\varepsilon(t)\|_{\hat{H}}^2 - \frac{1}{2} \|\hat{u}_1\|_{\hat{H}}^2 + \frac{1}{2} \hat{a}(\hat{u}_\varepsilon(t), \hat{u}_\varepsilon(t)) - \frac{1}{2} \hat{a}(\hat{u}_0, \hat{u}_0) + \int_0^t \hat{b}(\dot{\hat{u}}_\varepsilon, \dot{\hat{u}}_\varepsilon) ds \\ + \frac{1}{2\varepsilon} \|([\hat{u}_{\varepsilon N}(t)] - g)_+\|_{L^2(\mathcal{E})}^2 \leq \int_0^t \langle \hat{L}, \dot{\hat{u}}_\varepsilon \rangle ds, \quad \text{a.e. } t \in]0, T[. \end{aligned}$$

Thus, the following estimates on \hat{u}_ε hold, with $M > 0$ independent of ε :

$$\|\dot{\hat{u}}_\varepsilon\|_{\hat{H}} \leq M, \quad \|\hat{u}_\varepsilon\|_{\hat{V}} \leq M, \quad \|([\hat{u}_{\varepsilon N}] - g)_+\|_{L^2(\mathcal{E})} \leq M\sqrt{\varepsilon} \quad \text{a.e. } t \in]0, T[, \quad \int_0^T \|\dot{\hat{u}}_\varepsilon\|_{\hat{V}}^2 ds \leq M. \tag{6}$$

4. Passage to the limit and existence result

In (5), we replace \widehat{w} by $\widehat{v} + \dot{\widehat{u}}_\varepsilon - \widehat{u}_\varepsilon$, where $\widehat{v} \in L^\infty(0, T; \widehat{V}) \cap W^{1,2}(0, T; \widehat{H})$ is such that $\widehat{v} \in \widehat{K}$ for a.e. $t \in [0, T]$ and we integrate with respect to time variable t between 0 and T . We integrate by parts the acceleration term so that we obtain the following inequality:

$$\left\{ \begin{aligned} & \langle \dot{\widehat{u}}_\varepsilon(T), \widehat{v}(T) - \widehat{u}_\varepsilon(T) \rangle_{H^{-1/2}, H^{1/2}} - (\widehat{u}_1, \widehat{v}(0) - \widehat{u}_0) \\ & + \int_0^T \{ -(\dot{\widehat{u}}_\varepsilon, \widehat{v}) + \widehat{a}(\widehat{u}_\varepsilon, \widehat{v}) + \widehat{b}(\dot{\widehat{u}}_\varepsilon, \widehat{v}) + \widehat{J}(\widehat{u}_\varepsilon, \dot{\widehat{u}}_\varepsilon, \widehat{v} + \dot{\widehat{u}}_\varepsilon - \widehat{u}_\varepsilon) \} dt \\ & \geq \int_0^T \langle \widehat{L}, \widehat{v} - \widehat{u}_\varepsilon \rangle dt + \int_0^T \{ -\|\dot{\widehat{u}}_\varepsilon\|_{\widehat{H}}^2 + \widehat{a}(\widehat{u}_\varepsilon, \widehat{u}_\varepsilon) + \widehat{b}(\dot{\widehat{u}}_\varepsilon, \widehat{u}_\varepsilon) + \widehat{J}(\widehat{u}_\varepsilon, \dot{\widehat{u}}_\varepsilon, \widehat{u}_\varepsilon) \} dt \\ & \forall \widehat{v} \in L^\infty(0, T; \widehat{V}) \cap W^{1,2}(0, T; \widehat{H}), \widehat{v}(t) \in \widehat{K} \text{ a.e. } t \in]0, T[, \widehat{u}_\varepsilon(0) = \widehat{u}_0, \dot{\widehat{u}}_\varepsilon(0) = \widehat{u}_1. \end{aligned} \right. \tag{7}$$

We use (5) to obtain an estimate on acceleration: the term $\ddot{\widehat{u}}_\varepsilon$ is bounded in $L^2(0, T; [H^{-1}(\Omega^+)]^d \times [H^{-1}(\Omega^-)]^d)$. From (6) and the previous estimate, it follows that there exists $\widehat{u} = (\mathbf{u}^+, \mathbf{u}^-) \in W^{1,2}(0, T; \widehat{V}) \cap W^{1,\infty}(0, T; \widehat{H}) \cap W^{2,2}(0, T; [H^{-1}(\Omega^+)]^d \times [H^{-1}(\Omega^-)]^d)$ such that, up to a subsequence,

$$\begin{aligned} \dot{\widehat{u}}_\varepsilon &\rightharpoonup \widehat{u} \text{ weak } * \text{ in } L^\infty(0, T; \widehat{V}), \quad \dot{\widehat{u}}_\varepsilon \rightharpoonup \dot{\widehat{u}} \text{ in } L^2(0, T; \widehat{V}), \text{ weak } * \text{ in } L^\infty(0, T; \widehat{H}), \\ \ddot{\widehat{u}}_\varepsilon &\rightharpoonup \ddot{\widehat{u}} \text{ in } L^2(0, T; [H^{-1}(\Omega^+)]^d \times [H^{-1}(\Omega^-)]^d). \end{aligned} \tag{8}$$

Then, we can pass to the limit in linear elasticity and viscosity terms. For nonlinear terms, we need the following compactness result of Simon [12].

Lemma 4.1. *Let X, U and Y be three Banach spaces such that $X \subset U \subset Y$ with compact imbedding $X \rightarrow U$. Let \mathcal{F} be bounded in $L^p(0, T; X)$, where $1 \leq p < \infty$, and $\partial\mathcal{F}/\partial t = \{f; f \in \mathcal{F}\}$ be bounded in $L^1(0, T; Y)$. Then \mathcal{F} is relatively compact in $L^p(0, T; U)$.
Let \mathcal{F} be bounded in $L^\infty(0, T; X)$ and $\partial\mathcal{F}/\partial t$ be bounded in $L^r(0, T; Y)$, where $r > 1$. Then \mathcal{F} is relatively compact in $C([0, T]; U)$.*

We apply Lemma 4.1 with $\mathcal{F} = (\dot{\widehat{u}}_\varepsilon)_\varepsilon, X = \widehat{V}, U = \widehat{H}, Y = [H^{-1}(\Omega^+)]^d \times [H^{-1}(\Omega^-)]^d, \mathcal{F} = (\widehat{u}_\varepsilon)_\varepsilon, X = \widehat{V}, U = [H^{1/2}(\Omega^+)]^d \times [H^{1/2}(\Omega^-)]^d, Y = \widehat{H}$, and $\mathcal{F} = (\dot{\widehat{u}}_\varepsilon)_\varepsilon, X = \widehat{H}, U = [H^{-1/2}(\Omega^+)]^d \times [H^{-1/2}(\Omega^-)]^d, Y = [H^{-1}(\Omega^+)]^d \times [H^{-1}(\Omega^-)]^d$. Up to a subsequence, we obtain

$$\begin{aligned} \dot{\widehat{u}}_\varepsilon &\rightarrow \dot{\widehat{u}} \text{ in } L^2(0, T; \widehat{H}), \quad \widehat{u}_\varepsilon \rightarrow \widehat{u} \text{ in } C([0, T]; [H^{1/2}(\Omega^+)]^d \times [H^{1/2}(\Omega^-)]^d), \\ \dot{\widehat{u}}_\varepsilon &\rightarrow \dot{\widehat{u}} \text{ in } C([0, T]; [H^{-1/2}(\Omega^+)]^d \times [H^{-1/2}(\Omega^-)]^d). \end{aligned}$$

As $W^{1,2}(0, T; \widehat{V}) \subset C([0, T]; \widehat{V})$ and by (8), it follows that $(\widehat{u}_\varepsilon(t))_\varepsilon$ is bounded in $\widehat{V}, \forall t \in [0, T]$. By using a diagonal process as in [2], we can extract a subsequence still denoted by $(\widehat{u}_\varepsilon)_\varepsilon$, such that $\widehat{u}_\varepsilon(t) \rightharpoonup \widehat{u}(t)$ in $\widehat{V}, \forall t \in [0, T]$. Thus, based on classical weak lower semicontinuity arguments on $L^2(0, T; \widehat{V})$ and \widehat{V} , we obtain

$$\liminf_{\varepsilon \rightarrow 0} \int_0^T \widehat{a}(\widehat{u}_\varepsilon, \widehat{u}_\varepsilon) dt \geq \int_0^T \widehat{a}(\widehat{u}, \widehat{u}) dt, \quad \liminf_{\varepsilon \rightarrow 0} \int_0^T \widehat{b}(\dot{\widehat{u}}_\varepsilon, \widehat{u}_\varepsilon) dt \geq \int_0^T \widehat{b}(\dot{\widehat{u}}, \widehat{u}) dt.$$

Using (6), it follows that $([\widehat{u}_{\varepsilon N}](t))_\varepsilon$ is bounded in $H^{1/2}(\mathcal{E})$ independently of ε , for all $t \in [0, T]$, so that, up to a subsequence, we have $[\widehat{u}_{\varepsilon N}](t) \rightarrow [\widehat{u}_N](t)$ in $L^2(\mathcal{E}), \forall t \in [0, T]$. Then, from (6)₃, we derive

$$0 = \lim_{\varepsilon \rightarrow 0} \|([\widehat{u}_{\varepsilon N}] - g)_+\|_{L^2(\mathcal{E})} = \|([\widehat{u}_N] - g)_+\|_{L^2(\mathcal{E})}, \quad \text{a.e. } t \in]0, T[.$$

Hence $\hat{\mathbf{u}} \in \widehat{\mathbf{K}}$ a.e. $t \in [0, T]$. To pass to the limit in the friction term, we apply Lemma 4.1 to $\mathcal{F} = (\hat{\mathbf{u}}_\varepsilon)_\varepsilon$ and $\mathcal{F} = (\hat{\mathbf{u}}_\varepsilon)_\varepsilon$, where $X = \widehat{\mathbf{V}}$, $U = [H^{1-\delta}(\Omega^+)]^d \times [H^{1-\delta}(\Omega^-)]^d$, with $1/2 > \delta > 0$, $Y = [H^{-1}(\Omega^+)]^d \times [H^{-1}(\Omega^-)]^d$ so that we have $[\hat{\mathbf{u}}_{\varepsilon T}] \rightarrow [\hat{\mathbf{u}}_T]$ in $W^{1,2}(0, T; [L^2(\mathcal{E})]^d)$.

From the previous arguments and convergence property (2), we may pass to the lim inf in (7) to obtain

$$\left\{ \begin{array}{l} \langle \dot{\hat{\mathbf{u}}}(T), \hat{\mathbf{v}}(T) - \hat{\mathbf{u}}(T) \rangle_{H^{-1/2}, H^{1/2}} - (\hat{\mathbf{u}}_1, \hat{\mathbf{v}}(0) - \hat{\mathbf{u}}_0) \\ \quad + \int_0^T \{ -(\dot{\hat{\mathbf{u}}}, \dot{\hat{\mathbf{v}}}) + \hat{a}(\hat{\mathbf{u}}, \hat{\mathbf{v}}) + \hat{b}(\dot{\hat{\mathbf{u}}}, \dot{\hat{\mathbf{v}}}) + \hat{J}(\hat{\mathbf{u}}, \dot{\hat{\mathbf{u}}}, \hat{\mathbf{v}} + \dot{\hat{\mathbf{u}}} - \hat{\mathbf{u}}) \} dt \\ \geq \int_0^T \langle \hat{\mathbf{L}}, \hat{\mathbf{v}} - \hat{\mathbf{u}} \rangle dt + \int_0^T \{ -\|\dot{\hat{\mathbf{u}}}\|_{\widehat{\mathbf{H}}}^2 + \hat{a}(\hat{\mathbf{u}}, \hat{\mathbf{u}}) + \hat{b}(\dot{\hat{\mathbf{u}}}, \dot{\hat{\mathbf{u}}}) + \hat{J}(\hat{\mathbf{u}}, \dot{\hat{\mathbf{u}}}, \dot{\hat{\mathbf{u}}}) \} dt \\ \forall \hat{\mathbf{v}} \in L^\infty(0, T; \widehat{\mathbf{V}}) \cap W^{1,2}(0, T; \widehat{\mathbf{H}}), \hat{\mathbf{v}}(t) \in \widehat{\mathbf{K}} \text{ a.e. } t \in]0, T[. \end{array} \right. \quad (9)$$

Thus, \mathbf{u} defined on Ω by $\mathbf{u} = \mathbf{u}^+$ on Ω^+ , $\mathbf{u} = \mathbf{u}^-$ on Ω^- , is a solution of (3) and we have proved the following result.

Theorem 4.2. *Under the above assumptions there exists a subsequence of penalized solutions $(\mathbf{u}_\varepsilon)_\varepsilon$ which converges to a solution \mathbf{u} of dynamic unilateral contact problem with nonlocal friction P_1 .*

The existence of a solution to the purely elastic case (i.e., with $\mathbf{B} = \mathbf{0}$) in a nonsymmetric smooth domain remains an open problem to our knowledge.

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