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On the recovery of a manifold with boundary in $\mathbb{R}^n$

Philippe G. Ciarlet [a], Cristinel Mardare [b]

[a] *Department of Mathematics, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong*
[b] *Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, 4, place Jussieu, 75005 Paris, France*

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Abstract

If the Riemann curvature tensor associated with a smooth field $\boldsymbol{C}$ of positive-definite symmetric matrices of order n vanishes in a simply-connected open subset $\Omega \subset \mathbb{R}^n$, then $\boldsymbol{C}$ is the metric tensor field of a manifold isometrically immersed in $\mathbb{R}^n$.

In this Note, we first show how, under a mild smoothness assumption on the boundary of Ω, this classical result can be extended "up to the boundary". When Ω is bounded, we also establish the continuity of the manifold with boundary obtained in this fashion as a function of its metric tensor field, the topologies being those of the Banach spaces $\mathcal{C}^\ell(\overline{\Omega})$. ***To cite this article: P.G. Ciarlet, C. Mardare, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Sur la reconstruction d'une variété à bord dans $\mathbb{R}^n$. Si le tenseur de courbure de Riemann associé à un champ régulier $\boldsymbol{C}$ de matrices symétriques définies positives d'ordre n s'annule sur un ouvert $\Omega \subset \mathbb{R}^n$ simplement connexe, alors $\boldsymbol{C}$ est le champ de tenseurs métriques d'une variété plongée isométriquement dans $\mathbb{R}^n$.

Dans cette Note, on montre d'abord, moyennant une hypothèse peu restrictive sur la régularité de la frontière de Ω, comment ce résultat classique peut être étendu "jusqu'à la frontière". Lorsque Ω est borné, on établit aussi la continuité de la variété à bord ainsi obtenue en fonction de son champ de tenseurs métriques, les topologies étant celles des espaces de Banach $\mathcal{C}^\ell(\overline{\Omega})$. ***Pour citer cet article : P.G. Ciarlet, C. Mardare, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

Les notations non définies ici le sont dans la version anglaise.

Soit Ω un ouvert connexe et simplement connexe de $\mathbb{R}^n$, soit x_0 un point de Ω, et soit $\mathbb{S}^n$, resp. $\mathbb{S}^n_>$, l'ensemble de toutes les matrices symétriques, resp. symétriques définies positives, d'ordre n. Soit $\boldsymbol{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^n_>)$ une métrique riemannienne dont le tenseur de courbure de Riemann associé s'annule dans Ω. Autrement dit,

$$R^p_{\cdot ijk} := \partial_j \Gamma^p_{ik} - \partial_k \Gamma^p_{ij} + \Gamma^\ell_{ik} \Gamma^p_{j\ell} - \Gamma^\ell_{ij} \Gamma^p_{k\ell} = 0 \quad \text{dans } \Omega,$$

E-mail addresses: mapgc@cityu.edu.hk (P.G. Ciarlet), mardare@ann.jussieu.fr (C. Mardare).

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où

$$\Gamma_{ij}^{k} := \frac{1}{2} g^{k\ell}(\partial_i g_{j\ell} + \partial_j g_{\ell i} - \partial_\ell g_{ij}) \quad \text{et} \quad \big(g^{k\ell}\big) := (g_{ij})^{-1}.$$

Alors un théorème classique de géométrie différentielle (rappelé dans le Théorème 2.1) affirme qu'il existe une immersion $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbb{R}^n)$ et une seule telle que

$$\nabla\boldsymbol{\Theta}(x)^{\mathrm{T}} \nabla\boldsymbol{\Theta}(x) = \boldsymbol{C}(x) \quad \text{pour tout } x \in \Omega,$$
$$\boldsymbol{\Theta}(x_0) = \mathbf{0} \quad \text{et} \quad \nabla\boldsymbol{\Theta}(x_0) = \boldsymbol{C}(x_0)^{1/2}.$$

Notre *premier objectif* est d'étendre ce résultat « *jusqu'à la frontière* » de l'ouvert Ω, lorsque celui-ci vérifie ce que nous appelons la « *propriété géodésique* » (de fait, une hypothèse peu restrictive sur la régularité de la frontière $\partial\Omega$ de Ω ; voir Définition 1.2). Dans ce cas, on montre en effet (Théorème 2.2) que, si les fonctions $\partial^{\boldsymbol{\alpha}} g_{ij} \in \mathcal{C}^0(\Omega)$, $|\boldsymbol{\alpha}| \leqslant 2$, admettent des extensions continues sur $\overline{\Omega}$, les extensions des matrices (g_{ij}) restant définies positives sur l'ensemble $\overline{\Omega}$, alors les champs $\partial^{\boldsymbol{\alpha}}\boldsymbol{\Theta} \in \mathcal{C}^0(\Omega; \mathbb{R}^n)$, $|\boldsymbol{\alpha}| \leqslant 3$, admettent eux aussi des extensions continues sur $\overline{\Omega}$.

Soit $\mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n_>)$, resp. $\mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)$, l'ensemble formé des champs de matrices symétriques définies positives, resp. l'espace des champs de vecteurs, qui, avec toutes leurs dérivées partielles d'ordre $\leqslant 2$, resp. d'ordre $\leqslant 3$, admettent des extensions continues à $\overline{\Omega}$, les extensions des matrices restant définies positives sur $\overline{\Omega}$. Le Théorème 2.2 établit donc l'existence d'une application

$$\overline{\mathcal{F}}_0 : \boldsymbol{C} = (g_{ij}) \in \mathcal{C}^2_0\big(\overline{\Omega}; \mathbb{S}^n_>\big) \to \boldsymbol{\Theta} \in \mathcal{C}^3\big(\overline{\Omega}; \mathbb{R}^n\big),$$

où

$$\mathcal{C}^2_0\big(\overline{\Omega}; \mathbb{S}^n_>\big) := \big\{\boldsymbol{C} = (g_{ij}) \in \mathcal{C}^2\big(\overline{\Omega}; \mathbb{S}^n_>\big);\ R^{p}_{\cdot ijk} = 0 \text{ dans } \Omega\big\}.$$

Dans un autre travail (voir [6], et [7] pour les démonstrations détaillées), nous établissons aussi que, moyennant une hypothèse de régularité plus forte sur $\partial\Omega$, le résultat de prolongement ci-dessus combiné avec un théorème fondamental de Whitney conduit au résultat suivant de prolongement : Il existe un ouvert $\widetilde{\Omega}$ connexe de $\mathbb{R}^n$ contenant $\overline{\Omega}$ et un champ $\widetilde{\boldsymbol{C}} \in \mathcal{C}^2(\widetilde{\Omega}; \mathbb{S}^n_>)$ tels que $\widetilde{\boldsymbol{C}}$ prolonge $\boldsymbol{C}$ et le tenseur de courbure de Riemann associé à $\widetilde{\boldsymbol{C}}$ reste nul sur $\widetilde{\Omega}$.

Notre *second objectif* est de montrer que, lorsque l'ouvert Ω est *borné* et vérifie la *propriété géodésique*, alors *l'application* $\overline{\mathcal{F}}_0$ *est localement Lipschitz-continue*, les espaces $\mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n)$ et $\mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)$ étant munis de leurs normes usuelles (Théorème 3.1). On étend ainsi le résultat de continuité « dans le cas d'un ouvert » établi par Ciarlet et Laurent [5] pour les topologies de Fréchet des espaces $\mathcal{C}^2(\Omega; \mathbb{S}^n)$ et $\mathcal{C}^3(\Omega; \mathbb{R}^n)$.

Le présent travail trouve en partie son origine dans *la théorie de l'élasticité non-linéaire tri-dimensionnelle*. Comme l'avait déjà observé Antman [1], une autre approche de cette théorie consiste en effet à considérer le champ $\boldsymbol{C}$ de matrices, généralement appelées « tenseurs de Cauchy–Green » dans cette théorie, comme « l'inconnue principale », au lieu du champ $\boldsymbol{\Theta}$ de vecteurs, appelés « déformations » dans cette théorie. Effectivement, *la densité d'énergie d'un matériau hyperélastique est une fonction de* $\boldsymbol{C}$ (la forme de cette dépendance a d'ailleurs joué un rôle décisif dans les travaux de Ball [2], qui ont complètement renouvelé la théorie de l'elasticité). Cependant, la partie de l'énergie totale qui prend en compte les *forces appliquées*, de même que les *conditions aux limites*, restent des fonctions de $\boldsymbol{\Theta}$. D'où le besoin, dans cette approche, d'étudier *la dépendance d'un champ de déformations en fonction de son champ de tenseurs de Cauchy–Green, et cela, « jusqu'à la frontière »*. Naturellement, la réalisation d'un tel « programme » devra aussi inclure la considération de normes d'espaces de Sobolev, dans l'esprit des travaux de Kohn [12], ou de ceux, plus récents, de Friesecke, James et Müller [9] ou Reshetnyak [13].

Le même genre de questions, cette fois liées à la *théorie des coques non linéairement élastiques*, et donc relatives à la *théorie des surfaces dans* $\mathbb{R}^3$, sont abordées dans Ciarlet [3] et Ciarlet et Mardare [8].

On trouvera les démonstrations complètes des résultats de cette Note dans [7].

1. Preliminaries

Complete proofs of Theorems 2.2 and 3.1 are found in [7].

An integer $n \geqslant 2$ is chosen *once and for all* and Latin indices and exponents vary in the set $\{1, 2, \ldots, n\}$. The summation convention with respect to repeated indices and exponents is used. The notations $\mathbb{M}^n$, $\mathbb{S}^n$, and $\mathbb{S}^n_>$, respectively designate the sets of all square matrices, of all symmetric matrices, and of all positive-definite symmetric matrices, of order n. The spectral norm of a matrix $\boldsymbol{A} \in \mathbb{M}^n$ is denoted $|\boldsymbol{A}|$. In any metric space, the open ball with center x and radius $\delta > 0$ is denoted $B(x; \delta)$.

Let Ω be an open subset of $\mathbb{R}^n$. If $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \mathbb{R}^n)$ and $x \in \Omega$, the notation $\boldsymbol{\nabla\Theta}(x)$ stands for the matrix in $\mathbb{M}^n$ whose i-th column is the vector $\partial_i \boldsymbol{\Theta}(x) \in \mathbb{R}^n$. We also define the set

$$\mathcal{C}^2\big(\Omega; \mathbb{S}^n_>\big) := \big\{\boldsymbol{C} \in \mathcal{C}^2\big(\Omega; \mathbb{S}^n\big);\ \boldsymbol{C}(x) \in \mathbb{S}^n_> \text{ for all } x \in \Omega\big\}.$$

Central to this Note is the following notion of spaces of functions, vector fields, or matrix fields, "of class $\mathcal{C}^\ell$ up to the boundary of Ω".

Definition 1.1. Let Ω be an open subset of $\mathbb{R}^n$. For any integer $\ell \geqslant 1$, we define the space $\mathcal{C}^\ell(\overline{\Omega})$ as the space of all functions $f \in \mathcal{C}^\ell(\Omega)$ that, together with all their partial derivatives $\partial^{\boldsymbol{\alpha}} f$, $1 \leqslant |\boldsymbol{\alpha}| \leqslant \ell$, can be extended by continuity to $\overline{\Omega}$. Analogous definitions hold for the spaces $\mathcal{C}^\ell(\overline{\Omega}; \mathbb{R}^n)$, $\mathcal{C}^\ell(\overline{\Omega}; \mathbb{M}^n)$, and $\mathcal{C}^\ell(\overline{\Omega}; \mathbb{S}^n)$. All the continuous extensions appearing in such spaces will be identified by a bar, as exemplified in the definition of the following set:

$$\mathcal{C}^2\big(\overline{\Omega}; \mathbb{S}^n_>\big) := \big\{\boldsymbol{C} \in \mathcal{C}^2\big(\overline{\Omega}; \mathbb{S}^n\big);\ \overline{\boldsymbol{C}}(x) \in \mathbb{S}^n_> \quad \text{for all } x \in \overline{\Omega}\big\}.$$

Let Ω be a connected open subset of $\mathbb{R}^n$. Given two points $x, y \in \Omega$, a *path joining x to y in Ω* is any mapping $\boldsymbol{\gamma} \in \mathcal{C}^1([0,1]; \mathbb{R}^n)$ that satisfies $\boldsymbol{\gamma}(t) \in \Omega$ for all $t \in [0,1]$ and $\boldsymbol{\gamma}(0) = x$ and $\boldsymbol{\gamma}(1) = y$. Given a path $\boldsymbol{\gamma}$ joining x to y in Ω, its *length* is defined by $L(\boldsymbol{\gamma}) := \int_0^1 |\boldsymbol{\gamma}'(t)|\,\mathrm{d}t$. Let Ω be a connected open subset of $\mathbb{R}^n$. The *geodesic distance* between two points $x, y \in \Omega$ is defined by

$$d_\Omega(x, y) = \inf\big\{L(\boldsymbol{\gamma});\ \boldsymbol{\gamma} \text{ is a path joining } x \text{ to } y \text{ in } \Omega\big\}.$$

The results of this Note are established under a specific, but mild, regularity assumption on the boundary of an open subset of $\mathbb{R}^n$, according to the following definition:

Definition 1.2. An open subset Ω of $\mathbb{R}^n$ satisfies the *geodesic property* if it is connected and, given any point $x_0 \in \partial\Omega$ and any $\varepsilon > 0$, there exists $\delta = \delta(x_0, \varepsilon) > 0$ such that

$$d_\Omega(x, y) < \varepsilon \quad \text{for all } x, y \in \Omega \cap B(x_0; \delta).$$

Note that any connected open subset of $\mathbb{R}^n$ with a Lipschitz-continuous boundary possesses the geodesic property.

2. Recovery of a manifold with boundary from a prescribed metric tensor field

Let a Riemannian metric $(g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^n_>)$ be given over an open subset Ω of $\mathbb{R}^n$. The Christoffel symbols of the second kind associated with this metric are then defined by

$$\Gamma^k_{ij} := \frac{1}{2} g^{k\ell}(\partial_i g_{j\ell} + \partial_j g_{\ell i} - \partial_\ell g_{ij}), \quad \text{where } \big(g^{k\ell}\big) := (g_{ij})^{-1},$$

and the mixed components of its Riemann curvature tensor are defined by

$$R^p_{\cdot ijk} := \partial_j \Gamma^p_{ik} - \partial_k \Gamma^p_{ij} + \Gamma^\ell_{ik}\Gamma^p_{j\ell} - \Gamma^\ell_{ij}\Gamma^p_{k\ell}.$$

If this tensor vanishes in Ω and Ω is simply-connected, a classical result in differential geometry asserts that (g_{ij}) is the metric tensor field of a *manifold* $\boldsymbol{\Theta}(\Omega)$ that is isometrically immersed in $\mathbb{R}^n$. More precisely, we have (see, e.g., Ciarlet and Larsonneur [4, Theorem 2] for an elementary and self-contained proof):

Theorem 2.1. *Let Ω be a connected and simply-connected open subset of $\mathbb{R}^n$ and let a point $x_0 \in \Omega$ be given. Let a matrix field $\boldsymbol{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^n_>)$ be given that satisfies $R^p_{\cdot ijk} = 0$ in Ω. Then there exists one, and only one, immersion $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbb{R}^n)$ that satisfies*

$$\nabla\boldsymbol{\Theta}(x)^{\mathrm{T}}\nabla\boldsymbol{\Theta}(x) = \boldsymbol{C}(x) \quad \textit{for all } x \in \Omega,$$
$$\boldsymbol{\Theta}(x_0) = \mathbf{0} \quad \textit{and} \quad \nabla\boldsymbol{\Theta}(x_0) = \boldsymbol{C}(x_0)^{1/2}.$$

Remark 1. The additional conditions $\boldsymbol{\Theta}(x_0) = \mathbf{0}$ and $\nabla\boldsymbol{\Theta}(x_0) = \boldsymbol{C}(x_0)^{1/2}$ imply the uniqueness of the immersion $\boldsymbol{\Theta}$, which otherwise would be only determined up to isometries in $\mathbb{R}^n$. In fact, any additional conditions of the form $\boldsymbol{\Theta}(x_0) = \boldsymbol{a}_0$ and $\nabla\boldsymbol{\Theta}(x_0) = \boldsymbol{F}_0$, where $\boldsymbol{a}_0$ is any vector in $\mathbb{R}^n$ and $\boldsymbol{F}_0$ is any matrix in $\mathbb{M}^n$ that satisfies $\boldsymbol{F}_0^{\mathrm{T}}\boldsymbol{F}_0 = \boldsymbol{C}(x_0)$, likewise imply the uniqueness of the mapping $\boldsymbol{\Theta}$. The particular choice $\boldsymbol{F}_0 = \boldsymbol{C}(x_0)^{1/2}$ made here insures that the associated mapping $\boldsymbol{C}(x_0) \in \mathbb{S}^n_> \to \boldsymbol{F}_0 \in \mathbb{M}^n$ is smooth, a property that is used in the proof of Theorem 3.1. Another choice for the matrix $\boldsymbol{F}_0$ that fulfills the same criterion is the upper triangular matrix that arises in the Cholesky factorization of the matrix $\boldsymbol{C}(x_0)$.

The *first objective* of this Note is to show in the next theorem how a *manifold with boundary*, i.e., a subset of $\mathbb{R}^n$ of the form $\boldsymbol{\Theta}(\overline{\Omega})$, can be likewise recovered from a metric tensor field that, together with some partial derivatives, can be continuously extended to the closure $\overline{\Omega}$. In other words, we wish to extend the above existence and uniqueness result "up to the boundary of Ω".

Theorem 2.2. *Let Ω be a simply-connected open subset of $\mathbb{R}^n$ that satisfies the geodesic property* (see Definition 1.2) *and let a point $x_0 \in \Omega$ be given. Let there be given a matrix field $\boldsymbol{C} = (g_{ij}) \in \mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n_>)$* (in the sense of Definition 1.1) *that satisfies $R^p_{\cdot ijk} = 0$ in Ω. Then there exists one, and only one, mapping $\boldsymbol{\Theta} \in \mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)$* (again in the sense of Definition 1.1) *that satisfies* (the notations $\overline{\nabla\boldsymbol{\Theta}}$ and $\overline{\boldsymbol{C}}$ represent the continuous extensions of the fields $\nabla\boldsymbol{\Theta}$ and $\boldsymbol{C}$):

$$\overline{\nabla\boldsymbol{\Theta}}(x)^{\mathrm{T}}\overline{\nabla\boldsymbol{\Theta}}(x) = \overline{\boldsymbol{C}}(x) \quad \textit{for all } x \in \overline{\Omega},$$
$$\boldsymbol{\Theta}(x_0) = \mathbf{0} \quad \textit{and} \quad \nabla\boldsymbol{\Theta}(x_0) = \boldsymbol{C}(x_0)^{1/2}.$$

Sketch of proof. (i) Given any mapping $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbb{R}^n)$ that satisfies $\nabla\boldsymbol{\Theta}(x)^{\mathrm{T}}\nabla\boldsymbol{\Theta}(x) = \boldsymbol{C}(x)$ for all $x \in \Omega$ (such mappings exist by Theorem 2.1), let

$$\boldsymbol{F}(x) := \nabla\boldsymbol{\Theta}(x) \in \mathbb{M}^n \quad \text{and} \quad \boldsymbol{\Gamma}_i(x) := \big(\Gamma^k_{ij}(x)\big) \in \mathbb{M}^n \quad \text{for all } x \in \Omega,$$

with k as the row index and j as the column index. Then the matrix fields $\boldsymbol{F} \in \mathcal{C}^2(\Omega; \mathbb{M}^n)$ and $\boldsymbol{\Gamma}_i \in \mathcal{C}^1(\Omega; \mathbb{M}^n)$ defined in this fashion satisfy

$$\partial_i\boldsymbol{F}(x) = \boldsymbol{F}(x)\boldsymbol{\Gamma}_i(x) \quad \text{for all } x \in \Omega.$$

It is first established that the assumption $(g_{ij}) \in \mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n_>)$ implies that *the matrix fields $\boldsymbol{\Gamma}_i$ belong to the space $\mathcal{C}^1(\overline{\Omega}; \mathbb{M}^n)$* and that, *given any compact subset K of $\mathbb{R}^n$, then* $\sup_{x \in K \cap \Omega} |\boldsymbol{F}(x)| < \infty$.

Fix a point $x_0 \in \partial\Omega$ and let $K_0 = \overline{B(x_0; 1)}$. Then the above properties together imply that

$$c_0 := \Big(\sup_{x \in K_0 \cap \Omega} \big|\boldsymbol{F}(x)\big|\Big)\Bigg(\sup_{x \in K_0 \cap \Omega}\Big(\sum_i \big|\boldsymbol{\Gamma}_i(x)\big|^2\Big)^{1/2}\Bigg) < +\infty.$$

Let $\varepsilon > 0$ be given. Because Ω satisfies the *geodesic property*, there exists $\delta(\varepsilon) \in \,]0, \frac{1}{2}]$ such that, given any two points $x, y \in B(x_0; \delta(\varepsilon)) \cap \Omega$, there exists a path $\boldsymbol{\gamma} = (\gamma_i)$ joining x to y in Ω whose length satisfies $L(\boldsymbol{\gamma}) \leqslant \varepsilon / \max\{c_0, 2\}$.

Since $\partial_i \boldsymbol{F}(x) = \boldsymbol{F}(x)\boldsymbol{\Gamma}_i(x)$ for all $x \in \Omega$, the matrix field $\boldsymbol{Y} := \boldsymbol{F} \circ \boldsymbol{\gamma} \in \mathcal{C}^1([0,1]; \mathbb{M}^n)$ associated with any such path $\boldsymbol{\gamma}$ satisfies $\boldsymbol{Y}'(t) = \gamma_i'(t)\boldsymbol{Y}(t)\boldsymbol{\Gamma}_i(\boldsymbol{\gamma}(t))$ for all $0 \leqslant t \leqslant 1$. Expressing that $\boldsymbol{Y}(1) = \boldsymbol{Y}(0) + \int_0^1 \boldsymbol{Y}'(t)\,\mathrm{d}t$ yields, for any two points $x, y \in B(x_0; \delta(\varepsilon)) \cap \Omega$,

$$
\begin{aligned}
\left|\boldsymbol{F}(y) - \boldsymbol{F}(x)\right| = \left|\boldsymbol{Y}(1) - \boldsymbol{Y}(0)\right| &\leqslant \Big(\sup_{0 \leqslant t \leqslant 1} \left|\boldsymbol{F}(\boldsymbol{\gamma}(t))\right|\Big) \int_0^1 \left|\gamma_i'(t)\right| \left|\boldsymbol{\Gamma}_i(\boldsymbol{\gamma}(t))\right| \mathrm{d}t \\
&\leqslant \Big(\sup_{x \in K_0 \cap \Omega} \left|\boldsymbol{F}(x)\right|\Big) \bigg(\sup_{x \in K_0 \cap \Omega} \Big(\sum_i \left|\boldsymbol{\Gamma}_i(x)\right|^2\Big)^{1/2}\bigg) L(\boldsymbol{\gamma}) \leqslant \varepsilon.
\end{aligned}
$$

This inequality then implies that *the field $\boldsymbol{F} \in \mathcal{C}^2(\Omega; \mathbb{M}^n)$ can be extended to a field that is continuous on $\overline{\Omega}$.*

Since $\partial_i \boldsymbol{F} = \boldsymbol{F}\boldsymbol{\Gamma}_i$ in Ω and $\boldsymbol{\Gamma}_i \in \mathcal{C}^1(\overline{\Omega}; \mathbb{M}^n)$, each field $\partial_i \boldsymbol{F} \in \mathcal{C}^1(\Omega; \mathbb{M}^n)$ can be extended to a field that is continuous on $\overline{\Omega}$; hence $\boldsymbol{F} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{M}^n)$. Differentiating the relations $\partial_i \boldsymbol{F} = \boldsymbol{F}\boldsymbol{\Gamma}_i$ in Ω further shows that $\boldsymbol{F} \in \mathcal{C}^2(\overline{\Omega}; \mathbb{M}^n)$.

Given $x_0 \in \partial\Omega$, we proceed again as above, the number $\delta(\varepsilon) \in \,]0, \frac{1}{2}]$ being now chosen in such a way that $L(\boldsymbol{\gamma}) \leqslant \varepsilon / \max\{c_1, 2\}$, where $c_1 := \frac{1}{\sqrt{n}}(\sup_{x \in K_0 \cap \Omega} |\boldsymbol{F}(x)|)^{-1} < \infty$.

For each $x \in \Omega$, let $\boldsymbol{g}_i(x)$ denote the i-th column vector of the matrix $\boldsymbol{F}(x)$. The relations $\partial_i \boldsymbol{\Theta}(x) = \boldsymbol{g}_i(x)$ for all $x \in \Omega$ then imply that the vector field $\boldsymbol{y} := \boldsymbol{\Theta} \circ \boldsymbol{\gamma} \in \mathcal{C}^1([0,1]; \mathbb{R}^n)$ associated with each such path $\boldsymbol{\gamma}$ joining x to y in Ω satisfies $\boldsymbol{y}'(t) = \gamma_i'(t)\boldsymbol{g}_i(\boldsymbol{\gamma}(t))$ for all $0 \leqslant t \leqslant 1$, so that, for any two points $x, y \in B(x_0; \delta(\varepsilon)) \cap \Omega$,

$$
\begin{aligned}
\left|\boldsymbol{\Theta}(y) - \boldsymbol{\Theta}(x)\right| = \left|\boldsymbol{y}(1) - \boldsymbol{y}(0)\right| &\leqslant \int_0^1 \left|\gamma_i'(t)\boldsymbol{g}_i(\boldsymbol{\gamma}(t))\right| \mathrm{d}t \\
&\leqslant L(\boldsymbol{\gamma}) \sup_{x \in K_0 \cap \Omega} \Big(\sum_i \left|\boldsymbol{g}_i(x)\right|^2\Big)^{1/2} \leqslant \sqrt{n}\, L(\boldsymbol{\gamma}) \sup_{x \in K_0 \cap \Omega} \left|\boldsymbol{F}(x)\right| \leqslant \varepsilon.
\end{aligned}
$$

This inequality thus shows that *the field $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbb{R}^n)$ can be extended to a field that is continuous on $\overline{\Omega}$*. Differentiating the relations $\partial_i \boldsymbol{\Theta} = \boldsymbol{g}_i$ in Ω and noting that $\boldsymbol{g}_i \in \mathcal{C}^2(\overline{\Omega}; \mathbb{R}^n)$, we finally conclude that $\boldsymbol{\Theta} \in \mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)$. The uniqueness of $\boldsymbol{\Theta}$ is then easily established. □

We also establish elsewhere (see [6], and [7] for complete proofs) that, under a stronger regularity assumption on $\partial\Omega$, the above extension result combined with a fundamental theorem of Whitney leads to the following extension result: There exist a connected open subset $\widetilde{\Omega}$ of $\mathbb{R}^n$ containing $\overline{\Omega}$ and a field $\widetilde{\boldsymbol{C}} \in \mathcal{C}^2(\widetilde{\Omega}; \mathbb{S}^n_>)$ such that $\widetilde{\boldsymbol{C}}$ is an extension of $\boldsymbol{C}$ and the Riemann curvature tensor associated with $\widetilde{\boldsymbol{C}}$ still vanishes in $\widetilde{\Omega}$.

3. Continuity of a manifold with boundary as a function of its metric tensor

Let Ω be a simply-connected open subset of $\mathbb{R}^n$ that satisfies the geodesic property. Define the set

$$
\mathcal{C}_0^2(\overline{\Omega}; \mathbb{S}^n_>) := \left\{\boldsymbol{C} = (g_{ij}) \in \mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n_>);\ R^p_{\cdot ijk} = 0 \text{ in } \Omega\right\},
$$

and let again a point $x_0 \in \Omega$ be chosen *once and for all*. Then by Theorem 2.2, there exists a well-defined mapping

$$
\overline{\mathcal{F}}_0 : \mathcal{C}_0^2(\overline{\Omega}; \mathbb{S}^n_>) \to \mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)
$$

that associates with any matrix field $\boldsymbol{C} \in \mathcal{C}_0^2(\overline{\Omega}; \mathbb{S}_>^n)$ the unique mapping $\boldsymbol{\Theta} \in \mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)$ that satisfies $\overline{\boldsymbol{\nabla\Theta}}(x)^{\mathrm{T}}\overline{\boldsymbol{\nabla\Theta}}(x) = \overline{\boldsymbol{C}}(x)$ for all $x \in \overline{\Omega}$, $\boldsymbol{\Theta}(x_0) = \mathbf{0}$, and $\boldsymbol{\nabla\Theta}(x_0) = \boldsymbol{C}(x_0)^{1/2}$.

If in addition the set Ω is *bounded*, the spaces $\mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n)$ and $\mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)$ become Banach spaces and thus the set $\mathcal{C}_0^2(\overline{\Omega}; \mathbb{S}_>^n)$ becomes a metric space when it is equipped with the induced topology. So natural question arises: *Is the mapping $\overline{\mathcal{F}}_0$ continuous when the set $\mathcal{C}_0^2(\overline{\Omega}; \mathbb{S}_>^n)$ and the space $\mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)$ are equipped with these topologies*?

The *second objective* of this Note is to provide the following affirmative answer to this question.

Theorem 3.1. *Let Ω be a simply-connected and bounded open subset of $\mathbb{R}^n$ that satisfies the geodesic property, let the spaces $\mathcal{C}^\ell(\overline{\Omega}; \mathbb{M}^n)$ and $\mathcal{C}^\ell(\overline{\Omega}; \mathbb{R}^n)$, $\ell \geqslant 1$, be equipped with their usual norms, defined by*

$$\|\boldsymbol{F}\|_{\ell,\overline{\Omega}} = \sup_{\begin{cases} x\in\overline{\Omega} \\ |\boldsymbol{\alpha}|\leqslant \ell\end{cases}} \left|\overline{\partial^{\boldsymbol{\alpha}}\boldsymbol{F}}(x)\right| \quad \textit{for all } \boldsymbol{F} \in \mathcal{C}^\ell\big(\overline{\Omega}; \mathbb{M}^n\big),$$

$$\|\boldsymbol{\Theta}\|_{\ell,\overline{\Omega}} = \sup_{\begin{cases} x\in\overline{\Omega} \\ |\boldsymbol{\alpha}|\leqslant \ell\end{cases}} \left|\overline{\partial^{\boldsymbol{\alpha}}\boldsymbol{\Theta}}(x)\right| \quad \textit{for all } \boldsymbol{\Theta} \in \mathcal{C}^\ell\big(\overline{\Omega}; \mathbb{R}^n\big),$$

and let the set $\mathcal{C}_0^2(\overline{\Omega}; \mathbb{S}_>^n)$ be equipped with the metric induced by the norm $\|\cdot\|_{2,\overline{\Omega}}$. Then the mapping

$$\overline{\mathcal{F}}_0 : \mathcal{C}_0^2\big(\overline{\Omega}; \mathbb{S}_>^n\big) \to \mathcal{C}^3\big(\overline{\Omega}; \mathbb{R}^n\big)$$

is continuous. It is even locally Lipschitz-continuous over the set $\mathcal{C}_0^2(\overline{\Omega}; \mathbb{S}_>^n)$, in the sense that, given any matrix field $\widehat{\boldsymbol{C}} \in \mathcal{C}_0^2(\overline{\Omega}; \mathbb{S}_>^n)$, there exist constants $c(\widehat{\boldsymbol{C}}) > 0$ and $\delta(\widehat{\boldsymbol{C}}) > 0$ such that

$$\big\|\boldsymbol{\Theta} - \widetilde{\boldsymbol{\Theta}}\big\|_{3,\overline{\Omega}} \leqslant c\big(\widehat{\boldsymbol{C}}\big)\big\|\boldsymbol{C} - \widetilde{\boldsymbol{C}}\big\|_{2,\overline{\Omega}} \quad \textit{for all } \boldsymbol{C}, \widetilde{\boldsymbol{C}} \in \mathcal{C}_0^2\big(\overline{\Omega}; \mathbb{S}_>^n\big) \cap B\big(\widehat{\boldsymbol{C}}; \delta\big(\widehat{\boldsymbol{C}}\big)\big),$$

where $\boldsymbol{\Theta} := \overline{\mathcal{F}}_0(\boldsymbol{C})$, $\widetilde{\boldsymbol{\Theta}} := \overline{\mathcal{F}}_0(\widetilde{\boldsymbol{C}})$, and $B(\widehat{\boldsymbol{C}}; \delta(\widehat{\boldsymbol{C}}))$ denotes the open ball of center $\widehat{\boldsymbol{C}}$ and radius $\delta(\widehat{\boldsymbol{C}})$ in the space $\mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n)$.

Sketch of proof. (i) *Preliminaries.* The image $\boldsymbol{\Theta} = \overline{\mathcal{F}}_0(\boldsymbol{C}) \in \mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)$ of an arbitrary element $\boldsymbol{C} = (g_{ij}) \in \mathcal{C}_0^2(\overline{\Omega}; \mathbb{S}_>^n)$ is constructed in the following manner: *First*, the matrix fields $\boldsymbol{\Gamma}_i = (\Gamma_{ij}^k) \in \mathcal{C}^1(\overline{\Omega}; \mathbb{M}^n)$ are defined in Ω by letting

$$\Gamma_{ij}^k = \frac{1}{2} g^{k\ell}(\partial_i g_{j\ell} + \partial_j g_{\ell i} - \partial_\ell g_{ij}), \quad \text{where } \big(g^{k\ell}\big) = (g_{ij})^{-1},$$

and the matrix $\boldsymbol{C}(x_0)^{1/2} \in \mathbb{S}_>^n$ is defined as the unique square root of the matrix $\boldsymbol{C}(x_0)$.

Second, the matrix field $\boldsymbol{F} \in \mathcal{C}^2(\overline{\Omega}; \mathbb{M}^n)$ is defined as the unique one that satisfies

$$\partial_i \boldsymbol{F}(x) = \boldsymbol{F}(x)\boldsymbol{\Gamma}_i(x), \quad x \in \Omega, \quad \text{and} \quad \boldsymbol{F}(x_0) = \boldsymbol{C}(x_0)^{1/2}.$$

Third, the vector field $\boldsymbol{\Theta} \in \mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)$ is defined as the unique one that satisfies

$$\boldsymbol{\nabla\Theta}(x) = \boldsymbol{F}(x), \quad x \in \Omega, \quad \text{and} \quad \boldsymbol{\Theta}(x_0) = \mathbf{0}.$$

Accordingly, the strategy consists in establishing the local Lipschitz-continuity of each one of the above factor mapping separately, according to the following steps (the proofs of which are too technical to be sketched here):

(ii) Given any matrix field $\widehat{\boldsymbol{C}} \in \mathcal{C}_0^2(\overline{\Omega}; \mathbb{S}_>^n)$, there exist constants $c_1(\widehat{\boldsymbol{C}}) > 0$ and $\delta(\widehat{\boldsymbol{C}}) > 0$ such that

$$\big|\boldsymbol{C}(x_0)^{1/2} - \widetilde{\boldsymbol{C}}(x_0)^{1/2}\big| + \max_i \big\|\boldsymbol{\Gamma}_i - \widetilde{\boldsymbol{\Gamma}}_i\big\|_{1,\overline{\Omega}} \leqslant c_1\big(\widehat{\boldsymbol{C}}\big)\big\|\boldsymbol{C} - \widetilde{\boldsymbol{C}}\big\|_{2,\overline{\Omega}}$$

for all matrix fields $\boldsymbol{C}, \widetilde{\boldsymbol{C}} \in \mathcal{C}_0^2(\overline{\Omega}; \mathbb{S}_>^n) \cap B(\widehat{\boldsymbol{C}}; \delta(\widehat{\boldsymbol{C}}))$, where the matrix fields $\widetilde{\boldsymbol{\Gamma}}_i = (\widetilde{\Gamma}_{ij}^k) \in \mathcal{C}^1(\overline{\Omega}; \mathbb{M}^n)$ are defined in Ω by

$$\widetilde{\Gamma}_{ij}^k := \frac{1}{2}\widetilde{g}^{k\ell}(\partial_i \widetilde{g}_{j\ell} + \partial_j \widetilde{g}_{\ell i} - \partial_\ell \widetilde{g}_{ij}), \quad \text{where } (\widetilde{g}_{ij}) := \widetilde{\boldsymbol{C}} \text{ and } \left(\widetilde{g}^{k\ell}\right) := (\widetilde{g}_{ij})^{-1}.$$

(iii) The matrix fields $\boldsymbol{\Gamma}_i, \widetilde{\boldsymbol{\Gamma}}_i \in \mathcal{C}^1(\overline{\Omega}; \mathbb{M}^n)$ being defined in terms of the matrix fields $\boldsymbol{C}, \widetilde{\boldsymbol{C}} \in \mathcal{C}_0^2(\overline{\Omega}; \mathbb{S}_>^n)$ as in (i) and (ii), let the matrix fields $\boldsymbol{F}, \widetilde{\boldsymbol{F}} \in \mathcal{C}^2(\overline{\Omega}; \mathbb{M}^n)$ satisfy

$$\partial_i \boldsymbol{F}(x) = \boldsymbol{F}(x)\boldsymbol{\Gamma}_i(x) \quad \text{for all } x \in \Omega \quad \text{and} \quad \boldsymbol{F}(x_0) = \boldsymbol{C}(x_0)^{1/2},$$
$$\partial_i \widetilde{\boldsymbol{F}}(x) = \widetilde{\boldsymbol{F}}(x)\widetilde{\boldsymbol{\Gamma}}_i(x) \quad \text{for all } x \in \Omega \quad \text{and} \quad \widetilde{\boldsymbol{F}}(x_0) = \widetilde{\boldsymbol{C}}(x_0)^{1/2}.$$

Then, given any matrix field $\widehat{\boldsymbol{C}} \in \mathcal{C}_0^2(\overline{\Omega}; \mathbb{S}_>^n)$, there exists a constant $c_2(\widehat{\boldsymbol{C}}) > 0$ such that

$$\|\boldsymbol{F} - \widetilde{\boldsymbol{F}}\|_{2,\overline{\Omega}} \leqslant c_2(\widehat{\boldsymbol{C}})\left(\left|\boldsymbol{C}(x_0)^{1/2} - \widetilde{\boldsymbol{C}}(x_0)^{1/2}\right| + \max_i \|\boldsymbol{\Gamma}_i - \widetilde{\boldsymbol{\Gamma}}_i\|_{1,\overline{\Omega}}\right)$$

for all matrix fields $\boldsymbol{C}, \widetilde{\boldsymbol{C}} \in \mathcal{C}_0^2(\overline{\Omega}; \mathbb{S}_>^n) \cap B(\widehat{\boldsymbol{C}}; \delta(\widehat{\boldsymbol{C}}))$, where $\delta(\widehat{\boldsymbol{C}}) > 0$ is the constant found in (ii).

(iv) Let there be given matrix fields $\boldsymbol{F}, \widetilde{\boldsymbol{F}} \in \mathcal{C}^2(\overline{\Omega}; \mathbb{M}^n)$ and vector fields $\boldsymbol{\Theta}, \widetilde{\boldsymbol{\Theta}} \in \mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)$ that satisfy

$$\boldsymbol{\nabla}\boldsymbol{\Theta}(x) = \boldsymbol{F}(x) \quad \text{for all } x \in \Omega \quad \text{and} \quad \boldsymbol{\Theta}(x_0) = \boldsymbol{0},$$
$$\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}(x) = \widetilde{\boldsymbol{F}}(x) \quad \text{for all } x \in \Omega \quad \text{and} \quad \widetilde{\boldsymbol{\Theta}}(x_0) = \boldsymbol{0}.$$

Then there exists a constant $c_3 > 0$ independent of these fields such that

$$\|\boldsymbol{\Theta} - \widetilde{\boldsymbol{\Theta}}\|_{3,\overline{\Omega}} \leqslant c_3 \|\boldsymbol{F} - \widetilde{\boldsymbol{F}}\|_{2,\overline{\Omega}}. \qquad \square$$

The local Lipschitz-continuity of the mapping $\overline{\mathcal{F}}_0$ established in Theorem 3.1 is to be compared with the earlier estimates of John [10,11] and Kohn [12] and the recent "theorem on geometric rigidity" of Friesecke, James and Müller [9]. Such estimates are more powerful than those found here, in the sense that they are established for Sobolev norms. However, they only imply continuity at *rigid body deformations*, i.e., corresponding to $\boldsymbol{C} = \boldsymbol{I}$, whereas our estimates hold "at any Cauchy–Green tensor $\boldsymbol{C}$". Particularly relevant here is also the continuity result for "quasi-isometric mappings" recently established by Reshetnyak [13].

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