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# Kato's inequality when $\Delta u$ is a measure

Haïm Brezis [a,b], Augusto C. Ponce [a,b]

[a] *Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, BC 187, 4, pl. Jussieu, 75252 Paris cedex 05, France*
[b] *Rutgers University, Department of Math., Hill Center, Busch Campus, 110 Frelinghuysen Rd, Piscataway, NJ 08854, USA*

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**Abstract**

We extend the classical version of Kato's inequality in order to allow functions $u \in L^1_{\text{loc}}$ such that $\Delta u$ is a Radon measure. This inequality has been recently applied by Brezis, Marcus, and Ponce to study the existence of solutions of the nonlinear equation $-\Delta u + g(u) = \mu$, where $\mu$ is a measure and $g : \mathbb{R} \to \mathbb{R}$ is a nondecreasing continuous function. ***To cite this article: H. Brezis, A.C. Ponce, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***


**Résumé**

**L'inégalité de Kato lorsque $\Delta u$ est une mesure.** Nous étendons l'inégalité de Kato classique à des fonctions $u \in L^1_{\text{loc}}$ telles que $\Delta u$ est une mesure de Radon. Cette inégalité a été récemment utilisée par Brezis, Marcus et Ponce pour étudier l'existence de solutions de l'équation elliptique non linéaire $-\Delta u + g(u) = \mu$, où $\mu$ est une mesure et $g : \mathbb{R} \to \mathbb{R}$ est une fonction croissante et continue. ***Pour citer cet article : H. Brezis, A.C. Ponce, C. R. Acad. Sci. Paris, Ser. I 338 (2004).***


## Version française abrégée

Soient $N \geqslant 1$ et $\Omega \subset \mathbb{R}^N$ un ouvert borné quelconque. L'inégalité de Kato classique (voir [8]) affirme qu'étant donné $u \in L^1_{\text{loc}}(\Omega)$ tel que $\Delta u \in L^1_{\text{loc}}(\Omega)$, alors $\Delta u^+$ est une mesure de Radon et, de plus,

$$\Delta u^+ \geqslant \chi_{[u \geqslant 0]} \Delta u \quad \text{dans } \mathcal{D}'(\Omega). \tag{1}$$

Nous étendons (1) à des fonctions $u \in L^1_{\text{loc}}(\Omega)$ telles que $\Delta u \in \mathcal{M}(\Omega)$, où $\mathcal{M}(\Omega)$ désigne l'espace des mesures de Radon définies sur $\Omega$.

Rappelons que toute mesure $\mu \in \mathcal{M}(\Omega)$ peut être décomposée de façon unique comme une somme de deux mesures de Radon sur $\Omega$ (voir [7]) : $\mu = \mu_{\text{d}} + \mu_{\text{c}}$, avec

$$\mu_{\text{d}}(A) = 0 \quad \text{pour tout borélien } A \subset \Omega \text{ tel que } \operatorname{cap}(A) = 0,$$

*E-mail addresses:* brezis@ccr.jussieu.fr, brezis@math.rutgers.edu (H. Brezis), ponce@ann.jussieu.fr, augponce@math.rutgers.edu (A.C. Ponce).

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$$|\mu_c|(\Omega\backslash F)=0 \quad \text{pour un ensemble } F\subset\Omega \text{ fixé tel que } \operatorname{cap}(F)=0,$$

où cap dénote la capacité newtonienne ($W^{1,2}$). Les mesures $\mu_d$ et $\mu_c$ sont mutuellement singulières ; en particulier, $(\mu_d)^+=(\mu^+)_d$ et $(\mu_c)^+=(\mu^+)_c$.

Notre résultat principal est le suivant :

**Théorème 0.1.** *Soit $u\in L^1_{\mathrm{loc}}(\Omega)$ tel que $\Delta u\in\mathcal{M}(\Omega)$. Alors, $\Delta u^+\in\mathcal{M}(\Omega)$ et, de plus,*

$$(\Delta u^+)_d \geqslant \chi_{[u\geqslant 0]}(\Delta u)_d \quad \textit{dans } \Omega, \tag{2}$$

$$(-\Delta u^+)_c = (-\Delta u)^+_c \qquad \textit{dans } \Omega. \tag{3}$$

Le membre de droite dans (2) est bien défini, car la fonction $u$ est quasicontinue (voir [1] et aussi [4, Lemme 1]).

## 1. Introduction and main result

Let $N\geqslant 1$ and $\Omega\subset\mathbb{R}^N$ be a bounded open subset. The classical version of Kato's inequality (see [8]) asserts that given any function $u\in L^1_{\mathrm{loc}}(\Omega)$ such that $\Delta u\in L^1_{\mathrm{loc}}(\Omega)$, then $\Delta u^+$ is a Radon measure and the following holds:

$$\Delta u^+ \geqslant \chi_{[u\geqslant 0]}\Delta u \quad \text{in } \mathcal{D}'(\Omega). \tag{4}$$

Our main result (see Theorem 1.1 below) extends (4) to the case $\Delta u\in\mathcal{M}(\Omega)$, where $\mathcal{M}(\Omega)$ denotes the space of Radon measures on $\Omega$. In other words, $\mu\in\mathcal{M}(\Omega)$ if and only if, for every $\omega\subset\subset\Omega$, there exists $C_\omega>0$ such that $|\int_\Omega\varphi\,\mathrm{d}\mu|\leqslant C_\omega\|\varphi\|_\infty$, $\forall\varphi\in C_0^\infty(\omega)$.

We first recall that any $\mu\in\mathcal{M}(\Omega)$ can be uniquely decomposed as a sum of two Radon measures on $\Omega$ (see, e.g., [7]): $\mu=\mu_d+\mu_c$, where

$$\mu_d(A)=0 \quad \text{for any Borel measurable set } A\subset\Omega \text{ such that } \operatorname{cap}(A)=0,$$

$$|\mu_c|(\Omega\backslash F)=0 \quad \text{for some Borel measurable set } F\subset\Omega \text{ such that } \operatorname{cap}(F)=0.$$

Here, cap denotes the Newtonian ($W^{1,2}$) capacity of a set. We observe that $\mu_d$ and $\mu_c$ are singular with respect to each other. This decomposition is the analog of the classical Radon–Nikodym theorem, but with respect to cap. Clearly, $(\mu_d)^+=(\mu^+)_d$ and $(\mu_c)^+=(\mu^+)_c$.

Using the above notation, we can now state our main result:

**Theorem 1.1.** *Let $u\in L^1_{\mathrm{loc}}(\Omega)$ be such that $\Delta u\in\mathcal{M}(\Omega)$. Then, $\Delta u^+\in\mathcal{M}(\Omega)$, and the following holds*:

$$(\Delta u^+)_d \geqslant \chi_{[u\geqslant 0]}(\Delta u)_d \quad \textit{in } \Omega, \tag{5}$$

$$(-\Delta u^+)_c = (-\Delta u)^+_c \qquad \textit{in } \Omega. \tag{6}$$

Note that the right-hand side of (5) is well-defined because $u$ is quasicontinuous. More precisely, if $u\in L^1_{\mathrm{loc}}(\Omega)$ and $\Delta u\in\mathcal{M}(\Omega)$, then there exists $\tilde{u}:\Omega\to\mathbb{R}$ quasicontinuous such that $u=\tilde{u}$ a.e. in $\Omega$ (see [1] and also [4, Lemma 1]). In (5), we then identify $u$ with its quasicontinuous representative. It is easy to see that $\chi_{[u\geqslant 0]}$ is locally integrable in $\Omega$ with respect to the measure $|(\Delta u)_d|$.

The proof of (5) requires a theorem of Boccardo, Gallouët, and Orsina [2], which says that a Radon measure $\mu$ is diffuse (i.e., $\mu_c=0$) if and only if $\mu\in L^1_{\mathrm{loc}}(\Omega)+\Delta[H^1_{\mathrm{loc}}(\Omega)]$. Identity (6) relies on (and in fact is equivalent to) the 'inverse' maximum principle, recently established by Dupaigne and Ponce [6] (see Theorem 3.1 below).

Theorem 1.1 has been used by Brezis, Marcus and Ponce [5] to study the existence of solutions of the nonlinear equation $-\Delta u+g(u)=\mu$ where $\mu$ is a measure and $g$ is a nondecreasing continuous function.

An equivalent formulation of Theorem 1.1 is:

**Corollary 1.2.** *Let $u \in L^1_{\mathrm{loc}}(\Omega)$ be such that $\Delta u \in \mathcal{M}(\Omega)$. Then, $\Delta|u| \in \mathcal{M}(\Omega)$, and the following holds*:

$$(\Delta|u|)_{\mathrm{d}} \geqslant \operatorname{sgn}(u)(\Delta u)_{\mathrm{d}} \quad \text{in } \Omega, \tag{7}$$

$$(\Delta|u|)_{\mathrm{c}} = -|\Delta u|_{\mathrm{c}} \quad \text{in } \Omega. \tag{8}$$

Here, $\operatorname{sgn}(t) = 1$ for $t > 0$, $\operatorname{sgn}(t) = -1$ for $t < 0$, and $\operatorname{sgn}(0) = 0$.

**Remark 1.** A slight modification of the proof of Theorem 1.1 shows that

$$(\Delta u^+)_{\mathrm{d}} \geqslant \chi_{[u>0]}(\Delta u)_{\mathrm{d}} \quad \text{in } \Omega. \tag{9}$$

In other words, we can replace the set $[u \geqslant 0]$ in (5) by $[u > 0]$ and still get the same result.

Here is a simple consequence of (9):

**Corollary 1.3.** *Let $u \in L^1_{\mathrm{loc}}(\Omega)$ be such that $\Delta u \in \mathcal{M}(\Omega)$. If $u \geqslant 0$ a.e. in $\Omega$, then*

$$(\Delta u)_{\mathrm{d}} \geqslant 0 \quad \text{on the set } [u = 0]. \tag{10}$$

## 2. Proof of (5) in Theorem 1.1

We start with the following:

**Lemma 2.1.** *Assume $\mu \in \mathcal{M}(\Omega)$ is a diffuse measure with respect to* cap (*i.e.*, $\mu_{\mathrm{c}} = 0$). *Let $(v_n)$ be a sequence in $L^\infty(\Omega) \cap H^1(\Omega)$ such that $\|v_n\|_\infty \leqslant C$ and $v_n \rightharpoonup v$ weakly in $H^1$. Then,*

$$v_n \to v \quad \text{in } L^1_{\mathrm{loc}}(\Omega; \mathrm{d}\mu). \tag{11}$$

*Equivalently, there exists a subsequence $(v_{n_k})$ converging to $v$ $|\mu|$-a.e. in $\Omega$.*

**Proof.** Without loss of generality, we may assume that $|\mu|(\Omega) < \infty$. By Theorem 2.1 of Boccardo, Gallouët, and Orsina [2], we know that $\mu = f - \Delta g$ in $\mathcal{D}'(\Omega)$, for some $f \in L^1(\Omega)$ and $g \in H^1(\Omega)$. Using a standard density argument, we conclude that

$$\int_\Omega w\varphi \,\mathrm{d}\mu = \int_\Omega w\varphi f + \int_\Omega \nabla g \cdot \nabla(w\varphi), \quad \forall \varphi \in C_0^\infty(\Omega),\ \forall w \in L^\infty \cap H^1. \tag{12}$$

By assumption, the sequence $(|v_n - v|)$ is bounded in $H^1(\Omega)$ and, by Rellich's theorem, $|v_n - v| \to 0$ in $L^2(\Omega)$. Thus,

$$|v_n - v| \rightharpoonup 0 \quad \text{in } H^1. \tag{13}$$

Given $\varepsilon > 0$, let $\omega \subset\subset \Omega$ be such that $|\mu|(\Omega\backslash\omega) < \varepsilon$. We then fix $\varphi_0 \in C_0^\infty(\Omega)$ such that $0 \leqslant \varphi_0 \leqslant 1$ in $\Omega$ and $\varphi_0 = 1$ in $\omega$. Applying (12) with $w = |v_n - v|$ and $\varphi = \varphi_0$, we have

$$\begin{aligned}\int_\Omega |v_n - v| \,\mathrm{d}\mu &\leqslant \int_\omega |v_n - v| \,\mathrm{d}\mu + 2C|\mu|(\Omega\backslash\omega) \leqslant \int_\Omega |v_n - v|\varphi_0 \,\mathrm{d}\mu + 2C\varepsilon \\ &= \int_\Omega |v_n - v|\varphi_0 f + \int_\Omega \nabla g \cdot \nabla\big(|v_n - v|\varphi_0\big) + 2C\varepsilon.\end{aligned}$$

By (13), we know that $\int_\Omega \nabla g \cdot \nabla(|v_n - v|\varphi_0) \to 0$ as $n \to \infty$. Since $(v_n)$ is bounded in $L^\infty$ and $v_n \to v$ in $L^2$, we have $v_n \rightharpoonup v$ with respect to the weak* topology of $L^\infty$; thus, $\int_\Omega |v_n - v|\varphi_0 f \to 0$. We conclude that $\limsup_{n\to\infty} \int_\Omega |v_n - v| \,\mathrm{d}\mu \leqslant 2C\varepsilon$. Taking $\varepsilon > 0$ arbitrarily small, (11) follows. □

Given $k > 0$, we denote by $T_k : \mathbb{R} \to \mathbb{R}$ the truncation operator, i.e., $T_k(s) = s$ if $s \in [-k, k]$ and $T_k(s) = \operatorname{sgn}(s)k$ if $|s| > k$. Recall the following standard inequality (see, e.g., [4, Lemma 1]):

**Lemma 2.2.** *Assume $u \in L^1_{\mathrm{loc}}(\Omega)$ and $\Delta u \in \mathcal{M}(\Omega)$. Then, $T_k(u) \in H^1_{\mathrm{loc}}(\Omega)$, $\forall k > 0$; moreover, given $\omega \subset\subset \omega' \subset\subset \Omega$, there exists $C > 0$ such that*

$$\int_\omega \left|\nabla T_k(u)\right|^2 \leqslant k\left(\int_{\omega'} |\Delta u| + C\int_{\omega'} |u|\right). \tag{14}$$

Another ingredient used in the proof of (5) is our next result, which extends Lemma 2 in [3]:

**Proposition 2.1.** *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a $C^1$-convex function such that $0 \leqslant \Phi' \leqslant 1$ on $\mathbb{R}$. If $u \in L^1_{\mathrm{loc}}(\Omega)$ and $\Delta u \in \mathcal{M}(\Omega)$, then*

$$\Delta\Phi(u) \geqslant \Phi'(u)(\Delta u)_{\mathrm{d}} - (\Delta u)_{\mathrm{c}}^- \quad \text{in } \mathcal{D}'(\Omega). \tag{15}$$

**Proof.** Without loss of generality, we shall assume that $\Phi \in C^2$ and $\Phi''$ has compact support in $\mathbb{R}$. The general case can be easily deduced by approximation (note that since $\Phi$ is convex and $\Phi'$ is uniformly bounded, both limits $\Phi'(\pm\infty)$ exist and are finite). We may also assume that $u \in L^1(\Omega)$ and $\int_\Omega |\Delta u| < \infty$.

For every $x \in \Omega$, define $u_n(x) = \rho_n * u(x) = \int_\Omega \rho_n(x - y)u(y)\,\mathrm{d}y$, where $\rho_n$ is a family of radial mollifiers such that $\operatorname{supp}\rho_n \subset B_{1/n}$. Since $\Phi'' \geqslant 0$ in $\mathbb{R}$, we have

$$\Delta\Phi(u_n) = \Phi'(u_n)\Delta u_n + \Phi''(u_n)|\nabla u_n|^2 \geqslant \Phi'(u_n)\Delta u_n \quad \text{in } \Omega.$$

Let $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geqslant 0$. We multiply both sides of the inequality above by $\varphi$ and integrate by parts. For every $n \geqslant 1$ such that $d(\operatorname{supp}\varphi, \partial\Omega) > 1/n$, we have

$$\begin{aligned}\int_\Omega \Phi(u_n)\Delta\varphi \geqslant \int_\Omega \Phi'(u_n)\varphi\,\Delta u_n &= \int_\Omega \bigl\{\rho_n * \bigl[\Phi'(u_n)\varphi\bigr]\bigr\}\Delta u \\ &\geqslant \int_\Omega \bigl\{\rho_n * \bigl[\Phi'(u_n)\varphi\bigr]\bigr\}(\Delta u)_{\mathrm{d}} - \int_\Omega (\rho_n * \varphi)(\Delta u)_{\mathrm{c}}^-.\end{aligned}$$

Clearly,

$$\int_\Omega \Phi(u_n)\Delta\varphi \to \int_\Omega \Phi(u)\Delta\varphi \quad \text{and} \quad \int_\Omega (\rho_n * \varphi)(\Delta u)_{\mathrm{c}}^- \to \int_\Omega \varphi(\Delta u)_{\mathrm{c}}^-. \tag{16}$$

We now establish the following:

**Claim.** $\rho_n * [\Phi'(u_n)\varphi] \rightharpoonup \Phi'(u)\varphi$ *in* $H^1(\Omega)$.

In fact, since $\rho_n * [\Phi'(u_n)\varphi] \to \Phi'(u)\varphi$ in, say, $L^1(\Omega)$ and since $\varphi$ has compact support in $\Omega$, it suffices to show that $(\Phi'(u_n))$ is bounded in $H^1_{\mathrm{loc}}(\Omega)$. Let $M > 0$ be such that $\operatorname{supp}\Phi'' \subset [-M, M]$. Then,

$$\nabla\Phi'(u_n) = \Phi''(u_n)\nabla u_n = \Phi''(u_n)\nabla T_M(u_n) \quad \text{in } \Omega.$$

Let $\omega \subset\subset \omega' \subset\subset \Omega$. For $n \geqslant 1$ sufficiently large, it follows from (14) that

$$\int_\omega \left|\nabla \Phi'(u_n)\right|^2 \leqslant \|\Phi''\|_\infty \int_\omega \left|\nabla T_M(u_n)\right|^2 \leqslant CM\biggl(\int_{\omega'} |u_n| + \int_{\omega'} |\Delta u_n|\biggr) \leqslant CM\biggl(\int_\Omega |u| + \int_\Omega |\Delta u|\biggr),$$

for some constant $C > 0$ independent of $n$.

In view of the previous claim, we can now apply Lemma 2.1 above with $v_n = \rho_n * [\Phi'(u_n)\varphi]$ and $\mu = (\Delta u)_{\mathrm{d}}$ to conclude that

$$\int_\Omega \bigl\{\rho_n * \bigl[\Phi'(u_n)\varphi\bigr]\bigr\}(\Delta u)_{\mathrm{d}} \to \int_\Omega \Phi'(u)\varphi(\Delta u)_{\mathrm{d}}. \tag{17}$$

Combining (16) and (17) yields

$$\int_\Omega \Phi(u)\Delta\varphi \geqslant \int_\Omega \Phi'(u)\varphi(\Delta u)_{\mathrm{d}} - \int_\Omega \varphi\,(\Delta u)_{\mathrm{c}}^-, \quad \forall \varphi \in C_0^\infty(\Omega) \text{ with } \varphi \geqslant 0 \text{ in } \Omega,$$

which is precisely (15). □

**Proof of (5).** Let $(\Phi_n)$ be a sequence of smooth convex functions in $\mathbb{R}$ such that $\Phi_n(t) = t$ if $t \geqslant 0$ and $|\Phi_n(t)| \leqslant 1/n$ if $t < 0$. In particular, $0 \leqslant \Phi' \leqslant 1$ in $\mathbb{R}$. It follows from the previous proposition that

$$\Delta\Phi_n(u) \geqslant \Phi_n'(u)(\Delta u)_{\mathrm{d}} - (\Delta u)_{\mathrm{c}}^- \quad \text{in } \mathcal{D}'(\Omega).$$

As $n \to \infty$, we get

$$\Delta u^+ \geqslant \chi_{[u\geqslant 0]}(\Delta u)_{\mathrm{d}} - (\Delta u)_{\mathrm{c}}^- \quad \text{in } \mathcal{D}'(\Omega). \tag{18}$$

In particular, $\Delta u^+ \in \mathcal{M}(\Omega)$. Taking the diffuse part from both sides of (18), we conclude that (5) holds. □

## 3. Proof of (6) in Theorem 1.1

Identity (6) relies on the following:

**Theorem 3.1** ('Inverse' maximum principle [6]). *Let $u \in L^1_{\mathrm{loc}}(\Omega)$ be such that $\Delta u \in \mathcal{M}(\Omega)$. If $u \geqslant 0$ a.e. in $\Omega$, then*

$$(-\Delta u)_{\mathrm{c}} \geqslant 0 \quad \text{on } \Omega. \tag{19}$$

To complete the proof of Theorem 1.1, we now present:

**Proof of (6).** From the proof of (5), we already know that $\Delta u^+$ is a Radon measure on $\Omega$. Applying the 'inverse' maximum principle to $u^+$, we have $(-\Delta u^+)_{\mathrm{c}} \geqslant 0$ on $\Omega$. Since $u^+ - u \geqslant 0$ a.e. in $\Omega$, it also follows from Theorem 3.1 above that $(-\Delta u^+)_{\mathrm{c}} \geqslant (-\Delta u)_{\mathrm{c}}$ on $\Omega$. Thus,

$$(-\Delta u^+)_{\mathrm{c}} \geqslant (-\Delta u)_{\mathrm{c}}^+ \quad \text{on } \Omega,$$

which gives the '$\geqslant$' in (6). The reverse inequality just follows by taking the concentrated part from both sides of (18). In fact,

$$(-\Delta u^+)_{\mathrm{c}} \leqslant (\Delta u)_{\mathrm{c}}^- = (-\Delta u)_{\mathrm{c}}^+ \quad \text{on } \Omega.$$

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