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C. R. Acad. Sci. Paris, Ser. I 338 (2004) 397–402

COMPTES RENDUS MATHEMATIQUE

Differential Geometry

Computation of the Maslov index and the spectral flow via partial signatures

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Received 18 September 2003; accepted after revision 6 January 2004

Presented by Charles-Michel Marle

Abstract

Given a smooth Lagrangian path, both in the finite and in the infinite dimensional (Fredholm) case, we introduce the notion of partial signatures at each isolated intersection of the path with the Maslov cycle. For real-analytic paths, we give a formula for the computation of the Maslov index using the partial signatures; a similar formula holds for the spectral flow of real-analytic paths of Fredholm self-adjoint operators on real separable Hilbert spaces. As applications of the theory, we obtain a semi-Riemannian version of the Morse index theorem for geodesics with possibly conjugate endpoints, and we prove a bifurcation result at conjugate points along semi-Riemannian geodesics. ***To cite this article: R. Giambò et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Calcul de l'indice de Maslov et du flux spectral au moyen de signatures partielles. Etant donné un chemin régulier de lagrangiens, nous introduisons dans le cas de la dimension finie et le cas (Fredholm) de dimension infinie la notion de signatures partielles en chaque intersection isolée d'un tel chemin avec le cycle de Maslov. En utilisant les signatures partielles, nous donnerons une formule de calcul de l'indice de Maslov. Une formule semblabe vaut pour le flux spectral de chemins réel-analytiques d'opérateurs auto-adjoints de Fredholm sur des espaces de Hilbert réels et séparables. Comme application de la théorie, nous obtenons une version semi-Riemannienne du théorème de l'indice de Morse dans le cas de géodésiques avec des points initiaux conjugués. Enfin, nous démontrons un résultat de bifurcation en ces points conjugués le long des géodésiques semi-Riemanniennes. ***Pour citer cet article : R. Giambò et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

L'indice de Maslov d'un chemin lagrangien, dans le cas transversal, est donné par une somme algébrique des intersections du chemin avec le cycle de Maslov [10]. Dans ce travail nous introduisons la notion de signatures partielles [4,5,9] pour une intersection isolée d'un chemin régulière avec le cycle de Maslov, et nous donnons une formule pour le calcul de cet indice en termes de signatures partielles dans le cas d'intersections non transversales.

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2004.01.004

Soit V un espace vectoriel réel, $t \mapsto B(t)$ une courbe régulière (C^∞) de formes bilinéaires symétriques sur V et t_0 un temps isolé en lequel B est dégénérée. Un'*application racine* pour B en t_0 d'ordre $\mathrm{ord}(u) = k$ est une fonction régulière $t \mapsto u(t)$ telle que $t \mapsto B(t)u(t)$ a un zéro d'ordre k en $t = t_0$. Pour tout $k \geqslant 1$, soit W_k le sous espace de $\mathrm{Ker}(B(t_0))$ qui consiste en les vecteurs u_0 tels qu'il existe une fonction racine u pour B en t_0 avec $u(t_0) = u_0$ et $\mathrm{ord}(u) \geqslant k$. Nous définissons une forme bilinéaire symétrique B_k sur W_k en posons $B_k(u_0, v_0) = \frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}t^k}|_{t=t_0} B(t)(u(t), v(t))$, où u et v sont des fonctions racines pour B en $t = t_0$ d'ordre supérieur ou égal à k telles que $u(t_0) = u_0$ et $v(t_0) = v_0$. Un calcul simple montre que B_k est bien définie et $W_{k+1} = \mathrm{Ker}(B_k)$ pour tout $k \geqslant 1$. Nous désignons par $n_k^+(B, t_0)$, $n_k^-(B, t_0)$ et $\sigma_k(B, t_0)$ respectivement le co-indice, l'indice et la signature de B_k qu'on appelera collectivement les *signatures partielles* de B en t_0.

Soit maintenant (V, ω) un espace vectoriel symplectique, $\Lambda = \Lambda(V, \omega)$ la grassmannienne des lagrangiens de V avec $L_0 \in \Lambda$ un point fixe. Pour tout $m = 0, \dots, \frac{1}{2}\dim(V)$, on pose $\Lambda_m(L_0) = \{L \in \Lambda\colon \dim(L \cap L_0) = m\}$. L'ensemble $\Lambda_*(L_0) = \bigcup_{m \geqslant 1} \Lambda_m(L_0)$ est appelé le *cycle de Maslov avec le sommet en* L_0. Etant donné $L_1 \in \Lambda_0(L_0)$, pour tout $L \in \Lambda_0(L_1)$, on considère sur L_0 la forme bilinéaire symétrique $\varphi_{L_0,L_1}(L)$ définie par $\omega(T\cdot, \cdot)|_{L_0}$, où $T : L_0 \to L_1$ est l'unique opérateur linéaire dont le graphe est L. La famille des formes bilinéaires $\{\varphi_{L_0,L_1}\}$ constitue un atlas réel-analytique de Λ. Si $\gamma : [a, b] \to \Lambda$ est régulière et $t_0 \in [a, b]$ est une intersection isolée de γ avec $\Lambda_*(L_0)$, les signatures partielles de la courbe $\varphi_{L_0,L_1} \circ \gamma$ en t_0 ne dépendent pas du choix de L_1 et seront désignées par $n_k^+(\gamma, t_0; L_0)$, $n_k^-(\gamma, t_0; L_0)$ et $\sigma_k(\gamma, t_0; L_0)$. Notons par $\pi(\Lambda)$ le groupoïde fondamental de Λ muni de l'opération partielle de concaténation.

Proposition 0.1. *Il existe un unique homomorphisme de groupoïdes* (indice de Maslov) $\mu_{L_0} : \pi(\Lambda) \to \mathbb{Z}$ *tel que*

$$\mu_{L_0}([\gamma]) = n^+\big(\varphi_{L_0,L_1}(\gamma(b))\big) + \dim\big(\gamma(b) \cap L_0\big) - n^+\big(\varphi_{L_0,L_1}(\gamma(a))\big) - \dim\big(\gamma(a) \cap L_0\big)$$

pour $L_1 \in \Lambda_0(L_0)$ *et pour tout* $\gamma : [a, b] \to \Lambda$ *continue telle que* $\gamma([a, b]) \subset \Lambda_0(L_1)$.

Si $\gamma : [a, b] \to \Lambda$ *est réelle analytique et son image n'est pas entièrement contenue dans le cycle de Maslov, alors*

$$\mu_{L_0}([\gamma]) = \sum_{\substack{t_0 \in \gamma^{-1}(\Lambda_*(L_0)) \\ t_0 \in]a,b[}} \sum_k \sigma_{2k-1}(\gamma, t_0; L_0) + \dim\big(\gamma(b) \cap L_0\big) - \dim\big(\gamma(a) \cap L_0\big)$$
$$+ \sum_k n_k^+(\gamma, a; L_0) - \sum_k \big(n_{2k-1}^-(\gamma, b; L_0) + n_{2k}^+(\gamma, b; L_0)\big).$$

Pour tout couple $L_0, L_1 \in \Lambda$ et tout chemin continu $\gamma : [a, b] \to \Lambda$, la quantité $\mu_{L_1}([\gamma]) - \mu_{L_0}([\gamma])$ dépend seulement des extrémités $L_0' = \gamma(a)$ et $L_1' = \gamma(b)$ et sera désignée par $\mathfrak{q}(L_0, L_1; L_0', L_1')$. L'indice triple de Kashiwara $\tau(L_1, L_2, L_3)$ est donné par $\mathfrak{q}(L_1, L_2; L_3, L_1)$, d'autre part $\mathfrak{q}(L_0, L_1; L_0', L_1') = \tau(L_0, L_1, L_0') - \tau(L_0, L_1, L_1')$; il s'ensuit facilement que la fonction indice est univoquement déterminée par l'indice triple de Kashiwara [3,6]. La théorie s'étend intégralement au cas de Fredholm en dimension infinie, en remplaçant la différence des co-indices par le flux spectral des chemins d'opérateurs de Fredholm auto-adjoints.

Comme application, nous obtenons la version suivante du théorème de l'indice de Morse pour les géodésiques semi-Riemanniennes [8] avec des extrémités arbitraires. Soit (M, g) une variété semi-Riemannienne et $\theta : [a, b] \to M$ une géodésique ; l'évolution des données initiales par le biais du flux associé à l'équation de Jacobi produit une courbe de lagrangiens dont l'indice de Maslov sera noté par $i_{\mathrm{Maslov}}(\theta)$. L'*indice spectral* de θ, $i_{\mathrm{spectral}}(\theta)$, se définit comme le flux spectral du chemin de formes bilinéaires de Fredholm $]-\infty, 0] \ni \lambda \mapsto \int_a^b g(V', W') + g((R - \lambda)V, W)\,\mathrm{d}t$, où V, W sont des champs de classe H^1 nuls aux extrémités le long de θ et R est le tenseur de courbure de la connexion de Levi-Civita de g. L'*indice de Morse généralisé* de θ, $i_{\mathrm{Morse}}(\theta)$, se définit comme le flux spectral du chemin des formes bilinéaires de Fredholm $]a, b] \ni t \mapsto \int_a^t g(V', W') + g(RV, W)\,\mathrm{d}s$.

Proposition 0.2. $i_{\mathrm{Maslov}}(\theta) = i_{\mathrm{spectral}}(\theta) = i_{\mathrm{Morse}}(\theta)$.

1. Introduction

In the symplectic world, a recurrent notion is that of Maslov index, that appears naturally in many different contexts, especially in relation with solutions of Hamiltonian systems. The Maslov index is a $\mathbb{Z}$-valued homotopical invariant for continuous curves in the Lagrangian Grassmannian Λ that gives an algebraic measure of the intersections with the Maslov cycle [10]. Under generic circumstances, the intersections of a curve with the Maslov cycle occur at its regular part, and they are transversal (hence isolated). In this case the computation of the Maslov index is done via well established results obtained by differential topological methods, and its relations with the geometrical and analytical invariants of the variational problem are clear. Typically, the nontransversal case is studied by perturbative techniques, which allow us to extend to this case the results involving quantities that are stable by uniformly small perturbations. There are several reasons to develop a nonperturbative analysis of the nontransversal intersections, and that motivated the research exposed in this paper. In first place, perturbation arguments do not work properly when nontransversal intersections occur at the endpoints; namely, in this case arbitrarily small perturbations may destroy the intersection. Second, and more important, perturbative methods preserve global quantities, but destroy the information concerning each single intersection, which of course may be relevant in the problem under consideration. In semi-Riemannian geometry the presence of conjugate points have global geometrical implications on their own, and perturbing the data (i.e., the metric) would not be a very meaningful procedure. Along a semi-Riemannian geodesic, nontransversal intersections with the Maslov cycle (that may occur only if the metric is not positive definite) correspond to degenerate conjugate points; the presence of this kind of conjugate points is responsible for a series of new and interesting phenomena in the semi-Riemannian vs. the Riemannian world, hence it deserves a specific analysis.

2. Partial signatures, Maslov index and spectral flow

We introduce the notion of *partial signatures* [4,5,9] at an isolated, possibly nontransverse, intersection of a smooth curve in the (Fredholm) Lagrangian Grassmannian of a symplectic space with the Maslov cycle, and we give a formula for computing the Maslov index in terms of the partial signatures.

Let V be a real vector space, $t \mapsto B(t)$ a smooth (i.e., C^∞) curve of symmetric bilinear forms on V having an isolated degeneracy instant at $t = t_0$. A *root function for B of order k at t_0* is a smooth curve $t \mapsto u(t) \in V$ such that $t \mapsto B(t)u(t)$ has a zero of order k at $t = t_0$. For all $k \geqslant 1$, let W_k be the subspace of $\mathrm{Ker}(B(t_0))$ consisting of all vectors $u_0 \in V$ such that there exists a root function u for B at t_0 with $u(t_0) = u_0$ and $\mathrm{ord}(u) \geqslant k$. We define a symmetric bilinear form B_k on W_k by setting $B_k(u_0, v_0) = \frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}t^k}|_{t=t_0} B(t)(u(t), v(t))$, where u and v are root functions for B at t_0 with $\mathrm{ord}(u), \mathrm{ord}(v) \geqslant k$, $u(t_0) = u_0$ and $v(t_0) = v_0$. An easy computation shows that B_k is well defined, and that $W_{k+1} = \mathrm{Ker}(B_k)$ for all $k \geqslant 1$. We will denote by $n_k^-(B, t_0)$, $n_k^+(B, t_0)$ and $\sigma_k(B, t_0) = n_k^+(B, t_0) - n_k^-(B, t_0)$ respectively the index, the coindex and the signature of B_k; such integer numbers will be referred to collectively as the *partial signatures of B at $t = t_0$*. Let now (V, ω) be a symplectic vector space and let $\Lambda = \Lambda(V, \omega)$ denote the Lagrangian Grassmannian of V, with $L_0 \in \Lambda$ a fixed point; for all $m = 0, \dots, \frac{1}{2}\dim(V)$, set $\Lambda_m(L_0) = \{L \in \Lambda\colon \dim(L \cap l_0) = m\}$. The set $\Lambda_*(L_0) = \bigcup_{m \geqslant 1} \Lambda_m(L_0)$ is called the *Maslov cycle with vertex at L_0*, and it is an algebraic subvariety of Λ whose regular part is $\Lambda_1(L_0)$. Choose $L_1 \in \Lambda_0(L_0)$; for all $L \in \Lambda_0(L_1)$, let $\varphi_{L_0,L_1}(L)$ be the symmetric bilinear form on L_0 given by $\omega(T\cdot, \cdot)|_{L_0}$, where $T : L_0 \to L_1$ is the unique linear map whose graph is L. When L_0 and L_1 run in the set of transversal Lagrangians of V, the maps φ_{L_0,L_1} form a real-analytic atlas of charts of Λ. If $\gamma : [a, b] \to \Lambda$ is a smooth curve, and $t_0 \in [a, b]$ is an isolated intersection instant of γ with $\Lambda_*(L_0)$, then the partial signatures of the curve $t \mapsto \varphi_{L_0,L_1}(\gamma(t))$ at $t = t_0$ do not depend on the choice of L_1, and they will be denoted by $n_k^\pm(\gamma, t_0; L_0)$ and $\sigma_k(\gamma, t_0; L_0)$.

There are several definitions of Maslov index available in the literature, not always equivalent. Duistermaat's definition of Maslov index in [2] does not depend on the choice of L_0, but it is not additive by concatenation. A semi-integer valued Maslov index for arbitrary continuous Lagrangian paths, which has the two properties above has been introduced by Robbin and Salamon in [10] by considering first regular curves having transversal intersections with the Maslov cycle ("regular crossings", in the language of [10]) and then extending by homotopy invariance. The definition of Maslov index given by de Gosson (see [3]) is based on the notion of *Leray's index* for pairs in the universal covering of the Lagrangian Grassmannian Λ. Booss–Bavnbek and Furutani have given in [1] a functional analytical definition of Maslov index, both in the finite and infinite dimensional case, of a Fredholm Lagrangian paths. This is obtained by defining a one-parameter operator family of operators associated to the path, whose spectrum oscillates on the unit circle around $\mathrm{e}^{\mathrm{i}\pi}$ in the complex plane, and giving an appropriate algebraic count of the passages through a fixed gauge. The construction is done locally, and then patched together following Phillips' definition of spectral flow in [7]. We simplify the approach of [1] using a "van Kampen type theorem" for the fundamental groupoid of a topological space, that reduces the proof of the well-definiteness of the Maslov index to a simple local compatibility condition in the space of symmetric bilinear forms.

Let us denote by $\pi(\Lambda)$ the fundamental groupoid of Λ, endowed with the partial operation of concatenation, and by $[\gamma]$ the fixed-endpoint homotopy class of the continuous curve γ.

Proposition 2.1. *There exists a unique groupoid homomorphism $\mu_{L_0}:\pi(\Lambda)\to\mathbb{Z}$ such that*

$$\mu_{L_0}=n^+\big(\varphi_{L_0,L_1}\big(\gamma(b)\big)\big)+\dim\big(\gamma(b)\cap L_0\big)-n^+\big(\varphi_{L_0,L_1}\big(\gamma(a)\big)\big)-\dim\big(\gamma(a)\cap L_0\big)$$

for $L_1\in\Lambda_0(L_0)$ and for all $\gamma:[a,b]\to\Lambda$ continuous curve having image entirely contained in $\Lambda_0(L_1)$. If $\gamma:[a,b]\to\Lambda$ is real analytic and its image is not entirely contained in the Maslov cycle with vertex at L_0, then:

$$\mu_{L_0}\big([\gamma]\big)=\sum_{\substack{t_0\in\gamma^{-1}(\Lambda_*(L_0))\\ t_0\in]a,b[}}\sum_k\sigma_{2k-1}(\gamma,t_0;L_0)+\dim\big(\gamma(b)\cap L_0\big)-\dim\big(\gamma(a)\cap L_0\big)$$
$$+\sum_k n_k^+(\gamma,a;L_0)-\sum_k\big(n_{2k-1}^-(\gamma,b;L_0)+n_{2k}^+(\gamma,b;L_0)\big).$$

The map μ_{L_0} above will be called the *Maslov index*; in our notation, the definition of Maslov index of Robbin and Salamon in [10] for smooth curves $\gamma:[a,b]\to\Lambda$ having only transversal intersections with the Maslov cycle is given by:

$$\mu(\gamma,L_0)=\frac{1}{2}\sigma_1(\gamma,a;L_0)+\frac{1}{2}\sigma_1(\gamma,b;L_0)+\sum_{\substack{t_0\in\gamma^{-1}(\Lambda_*(L_0))\\ t_0\in]a,b[}}\sigma_1(\gamma,t_0;L_0).$$

The result of Proposition 2.1 holds true also in the case of curves in the Fredholm Lagrangian Grassmannian of an infinite dimensional symplectic Hilbert space, provided that the "difference of coindex" is replaced by the "spectral flow" of paths of self-adjoint Fredholm operators. Denote by $\mathcal{F}^{\mathrm{sa}}(\mathcal{H})$ the space of self-adjoint Fredholm operators on the real, separable Hilbert space $\mathcal{H}$; recall that, given a continuous path $[a,b]\ni t\mapsto\mathbf{S}(t)\in\mathcal{F}^{\mathrm{sa}}(\mathcal{H})$, the *spectral flow* $\mathrm{sf}(\mathbf{S},[a,b])$ is an integer number given by the net number of eigenvalues of $\mathbf{S}$ that pass through 0 as t runs from a to b. Let A and K be self-adjoint operators on a separable Hilbert space $\mathcal{H}$, with A invertible and K compact. Assume that $t_0\in\mathbb{R}\setminus\{0\}$ and that t_0^{-1} is in the spectrum of $-A^{-1}K$, so that t_0 is an isolated singularity of the affine path $t\mapsto\mathbf{S}(t)=A+tK$ in $\mathcal{F}^{\mathrm{sa}}(\mathcal{H})$. Then, for $\varepsilon>0$ small enough:

$$\mathrm{sf}\big(\mathbf{S},[t_0-\varepsilon,t_0]\big)=n^+(\mathcal{B}_2)-n^+(\mathcal{B}_1)+\dim\big(\mathrm{Ker}(A+t_0K)\big)=\mathring{n}^+(\mathcal{B}_2)-n^+(\mathcal{B}_1),$$
$$\mathrm{sf}\big(\mathbf{S},[t_0,t_0+\varepsilon]\big)=n^-(\mathcal{B}_1)-n^+(\mathcal{B}_2)-\dim\big(\mathrm{Ker}(A+t_0K)\big)=n^-(\mathcal{B}_1)-\mathring{n}^+(\mathcal{B}_2),$$

where $\mathcal{B}_1=\langle A\cdot,\cdot\rangle|_{\mathcal{H}_{t_0}}$, $\mathcal{B}_2=\langle(A+t_0K)\cdot,\cdot\rangle|_{\mathcal{H}_{t_0}}$, and $\mathcal{H}_{t_0}$ is the finite dimensional subspace of $\mathcal{H}$ given by

$$\mathcal{H}_{t_0}=\bigcup_{n\geqslant1}\mathrm{Ker}\bigg(A^{-1}K+\frac{1}{t_0}\mathrm{Id}\bigg)^n.$$

3. Kashiwara's and Hörmander's index

For every pair $L_0, L_1 \in \Lambda$ and every continuous curve $\gamma : [a,b] \to \Lambda$, the quantity $\mu_{L_1}(\gamma) - \mu_{L_0}(\gamma)$ only depends on the endpoints $L_0' = \gamma(a)$ and $L_1' = \gamma(b)$; we will denote this quantity by $\mathfrak{q}(L_0, L_1; L_0', L_1')$, this is known in the literature as the *Hörmander* four-fold index. The Hörmander's index $\mathfrak{q}$ has several symmetries, resembling those satisfied by the curvature tensor of a symmetric connection on a smooth manifold; observe that $\mathfrak{q}(L_0, L_1; L_0', L_1') = -\mathfrak{q}(L_0', L_1'; L_0, L_1)$, which, in turn, gives a cocycle identity for $\mathfrak{q}$. The *Kashiwara's triple index* $\tau : \Lambda \times \Lambda \times \Lambda \to \mathbb{Z}$ is related to $\mathfrak{q}$ by the equation $\tau(L_1, L_2, L_3) = \mathfrak{q}(L_1, L_2; L_3, L_1)$; likewise, $\mathfrak{q}(L_0, L_1; L_0', L_1') = \tau(L_0, L_1, L_0') - \tau(L_0, L_1, L_1')$. It follows that the Maslov index function is uniquely determined by the Kashiwara's index (see [3,6]):

$$\mu_{L_0}\big([\gamma]\big) = \sum_{i=1}^{M} \big[\tau\big(L_0, L_i, \gamma(t_{i-1})\big) - \tau\big(L_0, L_i, \gamma(t_i)\big)\big],$$

where $\gamma : [a,b] \to \Lambda$ is any continuous curve, $L_1, \ldots, L_M \in \Lambda$ and $t_0 = a < t_1 < \cdots < t_M = b$ are such that $\gamma([t_{i-1}, t_i]) \subset \Lambda_0(L_i)$ for all i.

4. The semi-Riemannian Morse index theorem

As an application of the theory of partial signatures, we obtain the following version of the semi-Riemannian Morse index theorem for geodesics, extending the result of [8] to the case of degenerate endpoints. Let (M,g) be a semi-Riemannian manifold and $\theta : [a,b] \to M$ a geodesic; the evolution of the initial data by the flow of the Jacobi equation along θ produces a smooth curve in the Lagrangian Grassmannian of a fixed symplectic space, whose Maslov index is denoted by $i_{\mathrm{Maslov}}(\theta)$. At each conjugate instant $t_0 \in [a,b]$ of θ, one defines the partial signatures $n_k^{\pm}(\theta, t_0)$ and $\sigma_k(\theta, t_0)$ as the partial signatures at t_0 of the corresponding curve in Λ. In the Riemannian or in the nonspacelike Lorentzian case, $n_1^{+}(\theta, t_0) = \sigma_1(\theta, t_0)$ equals the multiplicity of the conjugate instant, while $n_{k+1}^{+}(\theta, t_0) = n_k^{-}(\theta, t_0) = \sigma_{k+1}(\theta, t_0) = 0$ for all $k \geqslant 1$, and thus $i_{\mathrm{Maslov}}(\theta)$ equals the number of conjugate instants along θ in $]a,b]$ counted with multiplicity. More generally, denote by $\mathbb{J}_\theta$ the vector space of all Jacobi fields J along θ such that $J(a) = 0$, and by $\mathbb{J}_\theta[t_0] = \{J(t_0):\ J \in \mathbb{J}_\theta\} \subset T_{\theta(t_0)}M$. Then, $n_1^{\pm}(\theta, t_0)$ and $\sigma_1(\theta, t_0)$ are the signatures of the restriction of the metric g to the g-orthogonal space $\mathbb{J}_\theta[t_0]^{\perp}$; when such restriction is nondegenerate, then $n_k^{\pm}(\theta, t_0) = \sigma_k(\theta, t_0) = 0$ for all $k \geqslant 2$. If (M,g) is real analytic, in which case the set of conjugate instants along any compact geodesic portion is finite, then:

$$i_{\mathrm{Maslov}}(\theta) = \sum_{\substack{t_0 \text{ conjugate} \\ \text{instant in }]a,b[}} \sum_{k \geqslant 1} \sigma_{2k-1}(\theta, t_0) - \sum_{k \geqslant 1} \big[n_{2k-1}^{-}(\theta, b) + n_{2k}^{+}(\theta, b)\big] + n_\theta,$$

where n_θ is the *nullity of* θ, i.e., the multiplicity of the conjugate instant $t = b$. The *spectral index of* θ, denoted by $i_{\mathrm{spectral}}(\theta)$, is defined as the difference $n_\theta - \mathrm{sf}(I^\theta,]-\infty, 0])$, where I^θ is the real analytic path of Fredholm symmetric bilinear forms $I^\theta(\lambda)(V,W) = \int_a^b g(V', W') + g((R - \lambda)V, W)\,\mathrm{d}t$ defined on the space of all vector fields of Sobolev class H^1 along θ and vanishing at the endpoints. An explicit formula for the spectral index of θ can be given in terms of the generalized eigenspaces of the Jacobi differential operator along θ. The *generalized Morse index of* θ, denoted by $i_{\mathrm{Morse}}(\theta)$, is defined as the difference $n_\theta - \mathrm{sf}(S^\theta,]a,b])$, where $t \mapsto S^\theta(t)$ is the curve of Fredholm symmetric bilinear forms $S^\theta(t)(V,W) = \int_a^t g(V', W') + g(RV, W)\,\mathrm{d}s$.

Proposition 4.1 (Morse Index Theorem). $i_{\mathrm{Maslov}}(\theta) = i_{\mathrm{spectral}}(\theta) = i_{\mathrm{Morse}}(\theta)$.

Idea of the proof. The equality $i_{\mathrm{spectral}}(\theta) = i_{\mathrm{Morse}}(\theta)$ is obtained by a direct infinite dimensional homotopy argument, using the homotopy invariance of the spectral flow. A similar finite dimensional homotopy argument shows the equality of $i_{\mathrm{Maslov}}(\theta)$ and the Maslov index of the curve $]-\infty, 0] \ni \lambda \mapsto \ell(\lambda) = \Phi_\lambda(b)(L_0) \in \Lambda$, where $[a,b] \times \mathbb{R} \ni (t,\lambda) \mapsto \Phi_\lambda(t)$ is the resolvent of the Morse–Sturm equation $V'' = (R - \lambda)V$, seen as a first order system in $\mathbb{R}^{2n}$. The path $\ell(\lambda)$ in Λ is real-analytic; its intersections with the Maslov cycle occur at the eigenvalues of the Jacobi operator, which correspond precisely to the degeneracy instants of the path I^θ. The equality $i_{\mathrm{spectral}}(\theta) = i_{\mathrm{Maslov}}(\ell)$ is proven by a direct computation of the partial signatures at each nonpositive eigenvalue of the Jacobi operator. □

5. On the semi-Riemannian conjugate locus

The notion of partial signatures can be employed in the study of the conjugate locus in semi-Riemannian manifolds. Let (M,g) be a semi-Riemannian manifold, $\theta : [a,b] \to M$ be a geodesic in M and $t_0 \in\,]a,b[$. The point $\theta(t_0)$ is said to be a *bifurcation point along* θ if there exists a sequence $\theta_n : [a,b] \to M$ of geodesics in M and a sequence $(t_n)_{n\in\mathbb{N}} \subset\,]a,b[$ such that $\theta_n(a) = \theta(a)$ for all n, $\theta_n(t_n) = \theta(t_n)$ for all n, $\theta_n \to \theta$ as $n \to \infty$, $t_n \to t_0$ (and thus $\theta_n(t_n) \to \theta(t_0)$) as $n \to \infty$.

Proposition 5.1. *Let (M,g) be a real-analytic semi-Riemannian manifold, let $\theta : [a,b] \to M$ be a geodesic and let $t_0 \in\,]a,b[$ be a conjugate instant along θ.*

If $\sum_{k\geqslant 1} \sigma_{2k-1}(\theta, t_0) \neq 0$, then $\theta(t_0)$ is a bifurcation point along θ. Thus, the exponential map $\exp_{\theta(t_0)}$ is not injective on any neighborhood of $t_0\dot\theta(t_0)$.

In particular, every conjugate point along a Riemannian or a nonspacelike Lorentzian geodesic is a bifurcation point.

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